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CANONICAL EXPANSIONS OF INTEGERS IN REAL QUADRATIC FIELDS

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Dedicated to Professor Imre Kátai
on the occasion of his 65th birthday

Abstract. In [1, 2] a complete description was given for the location, number and structural properties of attractors generated by the discrete dynamics of the system (α, D), where α is an arbitrary integer in an imaginary quadratic field with norm $N \geq 2$ and D is the canonical digit set $\{0, 1, \ldots, N-1\}$. In this paper we shall extend some of these results to real quadratic fields.

1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Q}(\vartheta)$ its extension generated by an algebraic number ϑ. Let $\mathbb{Q}[\vartheta]$ denote the integers of $\mathbb{Q}(\vartheta)$. Let $\alpha \in \mathbb{Q}[\vartheta]$ be an algebraic integer for which α and all its conjugates have moduli greater than one. Let $|\mathrm{Norm}(\alpha)| \geq 2$ and let D be a complete residue system modulo α, for which $0 \in D$. The pair (α, D) is called a number system in $\mathbb{Q}[\vartheta]$ if for each $\gamma \in \mathbb{Q}[\vartheta]$ there exist an $m \in \mathbb{N}_0$ and $a_j \in D$ $(j = 0, 1, \ldots, m)$ such that $\gamma = a_0 + a_1\alpha + \ldots + a_m\alpha^m$. The uniqueness of the expansion follows from the fact that any two elements of D are incongruent modulo α. The system (α, D) can be used to represent all the integers $\gamma \in \mathbb{Q}[\vartheta]$ even if it is not a number system. Clearly, for each γ there exists a unique $a_j \in D$ such that

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$\alpha \mid \gamma - a_j$. Let $\gamma_1 = (\gamma - a_j)/\alpha$ and let us define the function $\Phi : \mathbb{Q}[\vartheta] \to \mathbb{Q}[\vartheta]$ by $\Phi(\gamma) = \gamma_1$. Let Φ^k denote the k-fold iterate of Φ, $\Phi^0(\gamma) = \gamma_0$. The sequence of integer vectors $\Phi^j(\gamma_0) = \gamma_j$ $(j = 1, 2, \ldots)$ is called the path of the dynamical system generated by Φ. An element $\pi \in \mathbb{Q}[\vartheta]$ is called periodic if there exists an $l \in \mathbb{N}$ such that $\Phi^l(\pi) = \pi$. The smallest such l is the length of period of π generated by Φ. Let $\mathcal{P}$ denote the set of all periodic elements. It is clear that $\pi \in \mathcal{P}$ if and only if there is an $l > 0$ such that

$$\pi = a_0 + a_1\alpha + \ldots + a_{l-1}\alpha^{l-1} + \pi\alpha^l, \quad a_j \in D. \tag{1}$$

The basin of attraction of $\pi \in \mathcal{P}$ consists of all $\gamma \in \mathbb{Q}[\vartheta]$ for which there exists a $j \in \mathbb{N}_0$ such that $\Phi^j(\gamma) = \pi$ and is denoted by $\mathcal{B}(\pi)$. Let $\mathcal{G}(\mathcal{P})$ be the directed graph defined on $\mathcal{P}$ by drawing an edge from $\pi \in \mathcal{P}$ to $\Phi(\pi)$. Then $\mathcal{G}(\mathcal{P})$ is a disjoint union of directed cycles, where loops are allowed. The concept of this kind of dynamics in algebraic number fields can be extended to the lattices of Euclidean spaces [3].

Consider the quadratic fields. Let α be a quadratic integer with minimal polynomial $x^2 + Ex + F$ and let D be the canonical digit set $\{0, 1, \ldots, |F| - 1\}$. Then (α, D) is a number system if and only if $F \geq 2$ and $-1 \leq E \leq F$ [4, 5, 6]. Moreover, in imaginary fields using canonical digit sets the set $\mathcal{G}(\mathcal{P})$ was completely described, i.e. the location, number and structure of periodic elements was fully determined. The aim of this paper is to describe the location of periodic elements in real quadratic fields.

2. Some lemmas

In the following we restrict our attention to the ring of integers in real quadratic fields.

Let $F \geq 2$ be a square-free integer. Let $\mathbb{Q}(\sqrt{F})$ be the real quadratic extension of $\mathbb{Q}$ generated by $\sqrt{F}$, I be the set of integers in $\mathbb{Q}(\sqrt{F})$. It is known, that if $F \not\equiv 1 \pmod 4$ then $\{1, \delta\}$, while for $F \equiv 1 \pmod 4$ $\{1, \omega\}$ is an integer basis in I, where $\delta = \sqrt{F}$, $\omega = (1 + \sqrt{F})/2$. The lattice generated by the basis $\{1, \delta\}$ will be called δ-lattice (denoted by Λ_δ) and the other one is ω-lattice (denoted by Λ_ω). For an arbitrary element $\beta = x + y\delta \in I$ or $\beta = x + y\omega \in I$ we shall denote its rational part x by $\mathcal{R}(\beta)$ and its irrational part y by $\mathcal{I}(\beta)$.

Let $\alpha_1 = a + b\delta$ and $\alpha_2 = a + b\omega$, $a, b \in \mathbb{Z}$, $b \neq 0$, $E = (F-1)/4$. In these cases the corresponding linear operators in $\mathbb{Z}^2$ are

$$M_1 = \begin{pmatrix} a & Fb \\ b & a \end{pmatrix} \qquad \text{and} \qquad M_2 = \begin{pmatrix} a & Eb \\ b & a+b \end{pmatrix}.$$

Clearly, $\det(M_1) = a^2 - Fb^2$ and $\det(M_2) = a^2 + ab - Eb^2$ and the first columns of the adjoint of the matrices M_1 and M_2 are $[a, -b]^T$ and $[a+b, -b]^T$, accordingly. Suppose that $(a,b) > 1$. If follows from [3] (Theorem 7) that in these cases the sets $\{0, 1, \ldots, |a^2 - Fb^2| - 1\}$ and $\{0, 1, \ldots, |a^2 + ab - Eb^2| - 1\}$ cannot be complete residue systems modulo M_1 and M_2, accordingly. Hence the following lemma holds.

Lemma 1. *For $\alpha \in \mathbb{Q}[\sqrt{F}]$ ($\alpha = a + b\delta$ or $\alpha = a + b\omega$) the set $D = \{0, 1, \ldots, |\alpha\overline{\alpha}| - 1\}$ is a complete residue system if and only if $(a,b) = 1$.*

Let $N = \alpha\overline{\alpha}$. Further we always consider canonical digit sets, hence we assume that $(a,b) = 1$, $b \neq 0$, $N \neq \pm 1$. The following lemma gives some information about the dynamics of the system (α, D).

Lemma 2. *Let $\alpha \in I, \pi \in \mathcal{P}$ in the system (α, D). (a) If $\gamma \in I$, $\gamma \equiv 0 \bmod \alpha, \gamma_1 = \gamma + d_1$, $\gamma_2 = \gamma + d_2$, $d_1, d_2 \in D$, then $\Phi(\gamma_1) = \Phi(\gamma_2)$. (b) If $\pi \in \mathcal{P}$ in (α, D) then $\overline{\pi} \in \mathcal{P}$ in $(\overline{\alpha}, D)$.*

Proof. (a) is obvious. Concerning (b) one can easily see that if $\pi = a_0 + a_1\alpha + \ldots + a_{l-1}\alpha^{l-1} + \pi\alpha^l$, $a_j \in D$ then $\overline{\pi} = a_0 + a_1\overline{\alpha} + \ldots + a_{l-1}\overline{\alpha}^{l-1} + \overline{\pi}\,\overline{\alpha}^l$, $a_j \in D$.

Since if $\alpha = a + b\delta$ then $\overline{\alpha} = a - b\delta$ and if $\alpha = a + b\omega$ then $\overline{\alpha} = a + b - b\omega$, therefore it is enough to examine the cases $b \geq 1$. Further we always assume that $b \geq 1$. In addition, we examine only those expansions for which $1-\alpha, 1-\overline{\alpha}$ are not units. The following lemma gives a lower estimation for the number of elements in D.

Lemma 3. *Consider the system (α, D), where $\alpha = a + b\delta$ or $\alpha = a + b\omega$. If $\alpha + 1$ is a unit then in the δ-lattice $|N| = 2|a|$ and in the ω-lattice $|N| = |2a + b|$. In all the other cases, in the δ-lattice $|N| \geq 2|a| + 1$ and in the ω-lattice $|N| \geq |2a + b| + 1$.*

Proof. Let $\alpha = a + b\delta$. We examine the following cases: (a) $\alpha, \overline{\alpha} > 0$. Then $a > 0$ and $0 < (\alpha-1)(\overline{\alpha}-1) = N - 2a + 1$, which is always greater than 1 since $1-\alpha$ is not a unit. (b) $\alpha, \overline{\alpha} < 0$. Then $a < 0$ and $0 < (\alpha+1)(\overline{\alpha}+1) = N + 2a + 1$. If $\alpha + 1$ is a unit then $N = 2|a|$. (c) $\operatorname{sgn}(a) = \operatorname{sgn}(\alpha) \neq \operatorname{sgn}(\overline{\alpha})$. Then $N < 0$ and $|\alpha| = |a + b\delta| = |2a - \overline{\alpha}| = |2a| + |\overline{\alpha}|$. Hence $|N| = |\alpha\overline{\alpha}| > |\alpha| > |2a| + 1$. (d) $\operatorname{sgn}(a) = \operatorname{sgn}(\overline{\alpha}) \neq \operatorname{sgn}(\alpha)$. Then $N < 0$ again and $|\overline{\alpha}| = |a - b\delta| = |2a - \alpha| = |2a| + |\alpha|$. Hence $|N| = |\alpha\overline{\alpha}| > |\overline{\alpha}| > |2a| + 1$.

Let $\alpha = a + b\omega$. Again, we have the following cases: (a) $\alpha, \overline{\alpha} > 0$. Then $2a+b > 0$ and $0 < (\alpha-1)(\overline{\alpha}-1) = N-2a-b+1$ which is always greater than 1, since $1-\alpha$ is not a unit. (b) $\alpha, \overline{\alpha} < 0$. Then $2a+b < 0$ and $0 < (\alpha+1)(\overline{\alpha}+1) = N+2a+b+1$. If $\alpha+1$ is a unit then $N = |2a+b|$. (c) $\operatorname{sgn}(2a+b) = \operatorname{sgn}(\alpha) \neq \operatorname{sgn}(\overline{\alpha})$. Then $N < 0$ and $|\alpha| = |a + b\omega| = |2a + b - \overline{\alpha}| = |2a + b| + |\overline{\alpha}|$. Hence $|N| = |\alpha\overline{\alpha}| > |\alpha| > |2a + b| + 1$. (d) $\operatorname{sgn}(2a + b) = \operatorname{sgn}(\overline{\alpha}) \neq \operatorname{sgn}(\alpha)$. Then $N < 0$ and $|\overline{\alpha}| = |a + b - b\omega| = |2a + b - \alpha| = |2a + b| + |\alpha|$. Hence $|N| = |\alpha\overline{\alpha}| > |\overline{\alpha}| > |2a + b| + 1$.

The following lemma shows that if a rational integer is a multiple of α then it is a multiple of $\alpha\overline{\alpha}$ as well.

Lemma 4. *Let $\alpha \in I$ ($\alpha = a + b\delta$ or $\alpha = a + b\omega$), $(a, b) = 1$, $f \in \mathbb{Z}$. If $\alpha \mid f$ then $N = \alpha\overline{\alpha} \mid f$.*

Proof. If $a + b\delta = \alpha \mid f$ then $(a + b\delta)(c + d\delta) = ac + bdF + (ad + bc)\delta = f$ for some $c, d \in \mathbb{Z}$. Since $(a, b) = 1$ therefore $c = ap$ and $d = -bp$ for some $p \in \mathbb{Z}$. Hence $a^2p - b^2Fp = f$, which means that $a^2 - b^2F \mid f$. If $a + b\omega = \alpha \mid f$ then $(a + b\omega)(c + d\omega) = ac + bdE + (ad + bc + bd)\omega = f$ for some $c, d \in \mathbb{Z}$. Again, since $(a, b) = 1$ therefore $c = (a + b)p$ and $d = -bp$ for some $p \in \mathbb{Z}$. It means that $a(a + b)p - Eb^2p = f$, by which the proof is finished.

3. Periodic elements with period length one

The following two lemmas show the number and location of periodic elements with period length one. First consider the δ-lattice and let $\alpha = a + b\delta,\ a, b \neq 0,\ (a, b) = 1$.

Lemma 5. *Let $L_\delta = \{\pi_j\}$ $(j = 0, \dots, k)$, where $\pi_j = (1-a+b\delta)j/(1-a, b)$ and $k = \lfloor(1 - a, b)(|N| - 1)/|(1 - a)^2 - b^2F|\rfloor$. If $\operatorname{sgn}\alpha = \operatorname{sgn}\overline{\alpha}$ then the periodic elements with period length one in the system (α, D) are the elements of L_δ, if $\operatorname{sgn}\alpha \neq \operatorname{sgn}\overline{\alpha}$ then the periodic elements with period length one are the elements of $-L_\delta$.*

Proof. It follows from (1) that $\pi \to \pi$ is a loop if and only if $\pi = d + \alpha\pi$ for some $d \in D$. It means that $(1 - \alpha)\pi = d \in D$, hence $\pi = d/(1 - \alpha) = d(1 - a)/((1 - a)^2 - b^2F) + \delta db/((1 - a)^2 - b^2F)$. Since $\pi \in I$ therefore $(1 - a)^2 - b^2F \mid d(1 - a, b)$. On the other hand $0 \leq d \leq |N| - 1$ and the signum of $N - 2a + 1$ is determined by $\operatorname{sgn}(\alpha)$ and $\operatorname{sgn}(\overline{\alpha})$. The proof is completed.

Consider now the ω-lattice. Let $\alpha = a + b\omega,\ b \neq 0,\ (a, b) = 1,\ N = a^2 + ab - b^2E,\ E = (F - 1)/4$. Using the same idea as by Lemma 5 the next result can easily be proved. We leave it to the reader.

Lemma 6. *Let* $L_\omega = \{\pi_j\}$ $(j = 0, \dots, k)$, *where* $\pi_j = (1-a-b+b\omega)j/(1-a-b, b)$ *and* $k = \lfloor (1-a-b, b)(|N|-1)/|(1-a)(1-a-b)-b^2E|\rfloor$. *If* $\operatorname{sgn}\alpha = \operatorname{sgn}\overline{\alpha}$ *then the periodic elements with period length one in the system* (α, D) *are the elements of* L_ω, *if* $\operatorname{sgn}\alpha \neq \operatorname{sgn}\overline{\alpha}$ *then the periodic elements with period length one are the elements of* $-L_\omega$.

4. Location of periodic elements

It is not hard to see that the method for determining the location of periodic elements, which was used in imaginary quadratic fields, cannot be used here, since α or $\overline{\alpha}$ can be very close to one in module, hence the convergence of M_1^{-i} or M_2^{-i} $(i = 1, 2, \dots, \infty)$ can be very slow. Our new method is based on the following idea: it is well-known (see e.g. in [3]) that all the periodic elements are inside the set $-H$, where $H = \left\{\sum_{i=1}^{\infty} M^{-i} d_i, d_i \in D\right\}$. First we prove a closed form for M^{-k} $(k \in \mathbb{N})$, then we shall compute the values $\min\{x \mid [x,y]^T \in H\}, \max\{x \mid [x,y]^T \in H\}$, $\min\{y \mid [x,y]^T \in H\}, \max\{y \mid [x,y]^T \in H\}$.

4.1. Expansions in Λ_δ

In the δ-lattice we shall work with the matrix

$$M_1 = \begin{pmatrix} a & Fb \\ b & a \end{pmatrix}.$$

Assertion 1. *With the notations introduced earlier we have*

$$M_1^{-k} = \frac{1}{2N^k} \begin{pmatrix} \alpha^k + \overline{\alpha}^k & -\sqrt{F}(\alpha^k - \overline{\alpha}^k) \\ -\frac{1}{\sqrt{F}}(\alpha^k - \overline{\alpha}^k) & \alpha^k + \overline{\alpha}^k \end{pmatrix}. \tag{2}$$

Proof. The proof is by induction. For the case $k = 1$ we have

$$M_1^{-1} = \frac{1}{2N} \begin{pmatrix} 2a & -2bF \\ -2b & 2a \end{pmatrix},$$

which is exactly the inverse of M_1. Suppose that (2) is true for k. We prove that (2) holds also for $k+1$. Since $M_1^{-(k+1)} = M_1^{-1} \cdot M_1^{-k}$ we have the following equations:

$$\begin{aligned}
\frac{a}{N}\left(\frac{\alpha^k+\overline{\alpha}^k}{2N^k}\right)+\frac{b}{N}\left(\frac{\alpha^k-\overline{\alpha}^k}{2N^k}\right)\sqrt{F} &= \frac{\alpha^{k+1}+\overline{\alpha}^{k+1}}{2N^{k+1}},\\
-\frac{Fb}{N}\left(\frac{\alpha^k+\overline{\alpha}^k}{2N^k}\right)-\frac{a}{N}\left(\frac{\alpha^k-\overline{\alpha}^k}{2N^k}\right)\sqrt{F} &= -\frac{\alpha^{k+1}-\overline{\alpha}^{k+1}}{2N^{k+1}}\sqrt{F},\\
-\frac{a}{N}\left(\frac{\alpha^k-\overline{\alpha}^k}{2\sqrt{F}N^k}\right)-\frac{b}{N}\left(\frac{\alpha^k+\overline{\alpha}^k}{2N^k}\right) &= -\frac{\alpha^{k+1}-\overline{\alpha}^{k+1}}{2\sqrt{F}N^{k+1}},\\
\frac{Fb}{N}\left(\frac{\alpha^k-\overline{\alpha}^k}{2\sqrt{F}N^k}\right)+\frac{a}{N}\left(\frac{\alpha^k+\overline{\alpha}^k}{2N^k}\right) &= \frac{\alpha^{k+1}+\overline{\alpha}^{k+1}}{2N^{k+1}}.
\end{aligned}$$

The proof is finished.

Let $\gamma_1 = \frac{1}{2}\sum\limits_{j=1}^{\infty}(\alpha^{2j}+\overline{\alpha}^{2j})/N^{2j}$, $\gamma_2 = \frac{1}{2}\sum\limits_{j=1}^{\infty}(\alpha^{2j-1}+\overline{\alpha}^{2j-1})/N^{2j-1}$, $\gamma_3 = = \frac{1}{2\sqrt{F}}\sum\limits_{j=1}^{\infty}(\alpha^{2j}-\overline{\alpha}^{2j})/N^{2j}$ and $\gamma_4 = \frac{1}{2\sqrt{F}}\sum\limits_{j=1}^{\infty}(\alpha^{2j-1}-\overline{\alpha}^{2j-1})/N^{2j-1}$. Further we examine some subcases. Computing the sums the computer algebra software Maple was used.

4.1.1. Case $\alpha, \overline{\alpha} > 0$

Since each member of γ_1 and γ_2 is positive, therefore

$$A_1 := \min\{x \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_1 := \max\{x \mid [x,y]^T \in H\} = \frac{1}{2}(|N|-1)\sum_{j=1}^{\infty}\frac{\alpha^j+\overline{\alpha}^j}{N^j} = \frac{(a-1)(\alpha\overline{\alpha}-1)}{(\alpha-1)(\overline{\alpha}-1)}.$$

Recall that $b > 0$, hence $\alpha > \overline{\alpha}$. Clearly, each member of γ_3 and γ_4 is positive, by which

$$A_2 := \min\{y \mid [x,y]^T \in H\} = -\frac{1}{2\sqrt{F}}(|N|-1)\sum_{j=1}^{\infty}\frac{\alpha^j-\overline{\alpha}^j}{N^j} = \frac{b(1-\alpha\overline{\alpha})}{(\alpha-1)(\overline{\alpha}-1)},$$

$$B_2 := \max\{y \mid [x,y]^T \in H\} = 0.$$

We got that if $\pi \in \mathcal{P}$ then $-B_1 \le \mathcal{R}(\pi) \le 0$ and $0 \le \mathcal{I}(\pi) \le -A_2$.

4.1.2. Case $\alpha, \overline{\alpha} < 0$

In this case each member of γ_1 is positive and each member of γ_2 is negative, hence

$$A_3 := \min\{x \mid [x,y]^T \in H\} = \gamma_2(|N|-1) = \frac{a(\alpha\overline{\alpha}-1)^2}{(\alpha^2-1)(\overline{\alpha}^2-1)},$$

$$B_3 := \max\{x \mid [x,y]^T \in H\} = \gamma_1(|N|-1) = \frac{(a^2+b^2F-1)(\alpha\overline{\alpha}-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

In the same way, each member of γ_3 is negative and each member of γ_4 is positive, hence

$$A_4 := \min\{y \mid [x,y]^T \in H\} = -\gamma_4(|N|-1) = \frac{b(1-\alpha^2\overline{\alpha}^2)}{(\alpha^2-1)(\overline{\alpha}^2-1)} \quad \text{and}$$

$$B_4 := \max\{y \mid [x,y]^T \in H\} = -\gamma_3(|N|-1) = \frac{2ab(1-\alpha\overline{\alpha})}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

We have that if $\pi \in \mathcal{P}$ then $-B_3 \le \mathcal{R}(\pi) \le -A_3$ and $-B_4 \le \mathcal{I}(\pi) \le -A_4$.

4.1.3. Case $\alpha > 0$, $\overline{\alpha} < 0$, $|\alpha| > |\overline{\alpha}|$

Clearly, each member of γ_1 is positive and each member of γ_2 is negative. We have that

$$A_5 := \min\{x \mid [x,y]^T \in H\} = \gamma_2(|N|-1) = \frac{a(\alpha\overline{\alpha}-1)(|\alpha\overline{\alpha}|-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)}, \quad \text{and}$$

$$B_5 := \max\{x \mid [x,y]^T \in H\} = \gamma_1(|N|-1) = \frac{(a^2+b^2F-1)(|\alpha\overline{\alpha}|-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

Since each member of γ_3 is positive and each member of γ_4 is negative, therefore

$$A_6 := \min\{y \mid [x,y]^T \in H\} = -\gamma_3(|N|-1) = \frac{2ab(1-|\alpha\overline{\alpha}|)}{(\alpha^2-1)(\overline{\alpha}^2-1)}, \quad \text{and}$$

$$B_6 := \max\{y \mid [x,y]^T \in H\} = -\gamma_4(|N|-1) = \frac{b(|\alpha\overline{\alpha}|-1)^2}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

We have that if $\pi \in \mathcal{P}$ then $-B_5 \le \mathcal{R}(\pi) \le -A_5$ and $-B_6 \le \mathcal{I}(\pi) \le -A_6$.

4.1.4. Case $\alpha > 0$, $\overline{\alpha} < 0$, $|\alpha| < |\overline{\alpha}|$

Observe that each member of γ_1 and γ_2 is positive. Hence

$$A_7 := \min\{x \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_7 := \max\{x \mid [x,y]^T \in H\} = \frac{1}{2}(|N|-1)\sum_{j=1}^{\infty}\frac{\alpha^j + \overline{\alpha}^j}{N^j} = \frac{(a-1)(|\alpha\overline{\alpha}|-1)}{(\alpha-1)(\overline{\alpha}-1)}.$$

Clearly, each member of γ_3 and γ_4 is negative, therefore

$$A_8 := \min\{y \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_8 := \max\{y \mid [x,y]^T \in H\} = \frac{1}{2\sqrt{F}}(|N|-1)\sum_{j=1}^{\infty}\frac{\alpha^j - \overline{\alpha}^j}{N^j} = \frac{b(|\alpha\overline{\alpha}|-1)}{(\alpha-1)(1-\overline{\alpha})}.$$

We have that if $\pi \in \mathcal{P}$ then $-B_7 \le \mathcal{R}(\pi) \le 0$ and $-B_8 \le \mathcal{I}(\pi) \le 0$.

4.1.5. Case $\alpha > 0$, $\overline{\alpha} < 0$, $|\alpha| = |\overline{\alpha}|$

In this case $a = 0$ and $b = 1$. Clearly, each member of γ_1 is positive and each member of γ_2 is 0. Therefore

$$A_9 := \min\{x \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_9 := \max\{x \mid [x,y]^T \in H\} = (|N|-1)\gamma_1 = (|N|-1)\sum_{i=1}^{\infty}\frac{1}{\alpha^{2j}} = \frac{|N|-1}{\alpha^2-1} = 1.$$

In the same way, each member of γ_3 is 0 and each member of γ_4 is negative. Hence,

$$A_{10} := \min\{y \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_{10} := \max\{y \mid [x,y]^T \in H\} = -(|N|-1)\gamma_4 = \frac{|N|-1}{\sqrt{F}}\sum_{i=1}^{\infty}\frac{1}{\alpha^{2j-1}} = \frac{\alpha}{\sqrt{F}} = 1.$$

It means that if $\pi \in \mathcal{P}$ then $-1 \le \mathcal{R}(\pi) \le 0$ and $-1 \le \mathcal{I}(\pi) \le 0$. Lemma 5 shows that $0 \in \mathcal{P}$ and $-1-\delta \in \mathcal{P}$. Let us see the expansion of -1. Since $-1 = -\alpha\overline{\alpha} - 1 + \alpha\overline{\alpha}$ and $\overline{\alpha} = 0 - \alpha$ therefore $\mathcal{G}(\mathcal{P}) = \{0 \to 0, -1-\delta \to \to -1-\delta, -1 \to \overline{\alpha} \to -1\}$.

4.2. Expansions in Λ_ω

In this case the appropriate matrix is

$$M_2 = \begin{pmatrix} a & Eb \\ b & a+b \end{pmatrix}.$$

It is easy to see that $\omega^2 = \omega + E$, $\overline{\omega} = (1-\sqrt{F})/2 = 1 - \omega$ and $\omega\overline{\omega} = -E$.

Assertion 2. *With the notations above we have*

$$M_2^{-k} = \frac{1}{\sqrt{F}N^k}\begin{pmatrix} \alpha^k\omega - \overline{\alpha}^k\overline{\omega} & -E(\alpha^k - \overline{\alpha}^k) \\ -(\alpha^k - \overline{\alpha}^k) & \overline{\alpha}^k\omega - \alpha^k\overline{\omega} \end{pmatrix}. \tag{3}$$

Proof. The proof is by induction. For the case $k = 1$ we have the following equations: $\alpha\omega - \overline{\alpha\alpha} = \sqrt{F}(a+b)$, $E(\alpha - \overline{\alpha}) = \sqrt{F}Eb$, $\overline{\alpha}\omega - \alpha\overline{\omega} = a\sqrt{F}$. Hence,

$$M_2^{-1} = \frac{1}{N}\begin{pmatrix} a+b & -Eb \\ -b & a \end{pmatrix},$$

which is exactly the inverse of M_2. Suppose that (3) is true for k. We prove that it holds also for $k+1$. Since $M_2^{-(k+1)} = M_2^{-1} \cdot M_2^{-k}$, we have the following equations:

$$(\alpha^k\omega - \overline{\alpha}^k\overline{\omega})(a+b) + Eb(\alpha^k - \overline{\alpha}^k) =$$
$$= \alpha^k(a\omega + b\omega^2) - \overline{\alpha}^k((a+b)\overline{\omega} + Eb) = \alpha^{k+1}\omega - \overline{\alpha}^{k+1}\overline{\omega},$$
$$-Eb(\alpha^k\omega - \overline{\alpha}^k\overline{\omega}) - Ea(\alpha^k - \overline{\alpha}^k) =$$
$$= -E(\alpha^k(b\omega + a) - \overline{\alpha}^k(b\overline{\omega} + a)) = -E(\alpha^{k+1} - \overline{\alpha}^{k+1}),$$
$$-(\alpha^k - \overline{\alpha}^k)(a+b) - b(\overline{\alpha}^k\omega - \alpha^k\overline{\omega}) =$$
$$= -\alpha^k(a + b - b\overline{\omega}) + \overline{\alpha}^k(a + b - b\omega) = -\alpha^{k+1} + \overline{\alpha}^{k+1},$$
$$Eb(\alpha^k - \overline{\alpha}^k) + a(\overline{\alpha}^k\omega - \alpha^k\overline{\omega}) =$$
$$= \overline{\alpha}^k(a\omega - Eb) - \alpha^k(a\overline{\omega} - Eb) = \overline{\alpha}^{k+1}\omega - \alpha^{k+1}\overline{\omega},$$

by which the proof is completed.

Let $\gamma_5 = \frac{1}{\sqrt{F}}\sum_{j=1}^{\infty}(\alpha^{2j}\omega^{2j} + \overline{\alpha}^{2j}\overline{\omega}^{2j})/N^{2j}$, $\gamma_6 = \frac{1}{\sqrt{F}}\sum_{j=1}^{\infty}(\alpha^{2j-1}\omega^{2j-1} + \overline{\alpha}^{2j-1}\overline{\omega}^{2j-1})/N^{2j-1}$, $\gamma_7 = \frac{1}{\sqrt{F}}\sum_{j=1}^{\infty}(\alpha^{2j} - \overline{\alpha}^{2j})/N^{2j}$ and $\gamma_8 = \frac{1}{\sqrt{F}}\sum_{j=1}^{\infty}(\alpha^{2j-1} - \overline{\alpha}^{2j-1})/N^{2j-1}$.

4.2.1. Case $\alpha, \overline{\alpha} > 0$

It is easy to see that in this case $\alpha > \overline{\alpha}$. Since each member of γ_5 and γ_6 is positive, therefore

$$A_{11} := \min\{x \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_{11} := \max\{x \mid [x,y]^T \in H\} = \frac{1}{\sqrt{F}}(|N|-1)\sum_{j=1}^{\infty}\frac{\alpha^j\omega^j - \overline{\alpha}^j\overline{\omega}^j}{N^j} =$$

$$= \frac{(a+b-1)(\alpha\overline{\alpha}-1)}{(\alpha-1)(\overline{\alpha}-1)}.$$

Clearly, each member of γ_7 and γ_8 is positive, by which

$$A_{12} := \min\{y \mid [x,y]^T \in H\} = -\frac{1}{\sqrt{F}}(|N|-1)\sum_{j=1}^{\infty}\frac{\alpha^j - \overline{\alpha}^j}{N^j} =$$

$$= \frac{b(1-\alpha\overline{\alpha})}{(\alpha-1)(\overline{\alpha}-1)},$$

$$B_{12} := \max\{y \mid [x,y]^T \in H\} = 0.$$

We got that if $\pi \in \mathcal{P}$ then $-B_{11} \le \mathcal{R}(\pi) \le 0$ and $0 \le \mathcal{I}(\pi) \le -A_{12}$.

4.2.2. Case $\alpha, \overline{\alpha} < 0$

In this case each member of γ_5 is positive and each member of γ_6 is negative, hence

$$A_{13} := \min\{x \mid [x,y]^T \in H\} = \gamma_6(|N|-1) = \frac{(\alpha\overline{\alpha}(a+b)-a)(\alpha\overline{\alpha}-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)},$$

$$B_{13} := \max\{x \mid [x,y]^T \in H\} = \gamma_5(|N|-1) = \frac{((a+b)^2+b^2E-1)(\alpha\overline{\alpha}-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

In the same way, each member of γ_7 is negative and each member of γ_8 is positive, hence

$$A_{14} := \min\{y \mid [x,y]^T \in H\} = -\gamma_8(|N|-1) = \frac{b(1-\alpha^2\overline{\alpha}^2)}{(\alpha^2-1)(\overline{\alpha}^2-1)},$$

$$B_{14} := \max\{y \mid [x,y]^T \in H\} = -\gamma_7(|N|-1) = \frac{b(2a+b)(1-\alpha\overline{\alpha})}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

We have that if $\pi \in \mathcal{P}$ then $-B_{13} \le \mathcal{R}(\pi) \le -A_{13}$ and $-B_{14} \le \mathcal{I}(\pi) \le -A_{14}$.

4.2.3. Case $\alpha > 0$, $\overline{\alpha} < 0$, $|\alpha| > |\overline{\alpha}|$

Clearly, each member of γ_5 is positive and each member of γ_6 is negative. We have that

$$A_{15} := \min\{x \mid [x,y]^T \in H\} = \gamma_6(|N|-1) = \frac{(\alpha\overline{\alpha}(a+b)-a)(|\alpha\overline{\alpha}|-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)},$$

$$B_{15} := \max\{x \mid [x,y]^T \in H\} = \gamma_5(|N|-1) = \frac{((a+b)^2+b^2E-1)(|\alpha\overline{\alpha}|-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

Since each member of γ_7 is positive and each member of γ_8 is negative, therefore

$$A_{16} := \min\{y \mid [x,y]^T \in H\} = -\gamma_7(|N|-1) = \frac{b(2a+b)(1-|\alpha\overline{\alpha}|)}{(\alpha^2-1)(\overline{\alpha}^2-1)},$$

$$B_{16} := \max\{y \mid [x,y]^T \in H\} = -\gamma_8(|N|-1) = \frac{b(|\alpha\overline{\alpha}|-1)^2}{(\alpha^2-1)(\overline{\alpha}^2-1)}.$$

We have that if $\pi \in \mathcal{P}$ then $-B_{15} \le \mathcal{R}(\pi) \le -A_{15}$ and $-B_{16} \le \mathcal{I}(\pi) \le -A_{16}$.

4.2.4. Case $\alpha > 0$, $\overline{\alpha} < 0$, $|\alpha| < |\overline{\alpha}|$

Obviously, each member of γ_5 is positive. Regarding γ_6 we have the following possibilities. Suppose that $\alpha\omega - \overline{\alpha\omega} \ge 0$. Then there is an odd integer $C \in \mathbb{N}$ for which

$$\frac{\omega}{\overline{\omega}} \le \left(\frac{\overline{\alpha}}{\alpha}\right)^C, \quad \text{but} \quad \frac{\omega}{\overline{\omega}} > \left(\frac{\overline{\alpha}}{\alpha}\right)^{C+2}.$$

Clearly, $\alpha^k\omega - \overline{\alpha}^k\overline{\omega} \ge 0$ for every odd k, $1 \le k \le C$ and $\alpha^k\omega - \overline{\alpha}^k\overline{\omega} < 0$ for every odd k, $k \ge C+2$. Hence,

$$A_{17} := \min\{x \mid [x,y]^T \in H\} = (|N|-1)\left(\sum_{j=1}^{(C+1)/2} \frac{\alpha^{2j-1}\omega - \overline{\alpha}^{2j-1}\overline{\omega}}{N^{2j-1}}\right) =$$

$$= \frac{(|\alpha\overline{\alpha}|-1)((1-\alpha^2)(1/\overline{\alpha}^C-\overline{\alpha})\omega-(1-\overline{\alpha}^2)(1/\alpha^C-\alpha)\overline{\omega})}{(\alpha^2-1)(\overline{\alpha}^2-1)\sqrt{F}} \quad \text{and}$$

$$B_{17} := \max\{x \mid [x,y]^T \in H\} = (|N|-1)(\gamma_5 + \sum_{j=(C+3)/2}^{\infty} \frac{\alpha^{2j-1}\omega - \overline{\alpha}^{2j-1}\overline{\omega}}{N^{2j-1}}) =$$

$$= \frac{((a+b)^2 + b^2E - 1)(|\alpha\overline{\alpha}| - 1)}{(\alpha^2-1)(\overline{\alpha}^2-1)} + \frac{((1-\alpha^2)\omega/\overline{\alpha}^C + (\overline{\alpha}^2-1)\overline{\omega}/\alpha^C)(|\alpha\overline{\alpha}|-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)\sqrt{F}}.$$

If $\alpha\omega < \overline{\alpha\omega}$ then each member of γ_6 is positive. Therefore

$$A_{17} := \min\{x \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_{17} := \max\{x \mid [x,y]^T \in H\} = \frac{(a+b-1)(|\alpha\overline{\alpha}|-1)}{(\alpha-1)(\overline{\alpha}-1)}.$$

Clearly, each member of γ_7 and γ_8 is negative, by which

$$A_{18} := \min\{y \mid [x,y]^T \in H\} = 0, \quad \text{and}$$

$$B_{18} := \max\{y \mid [x,y]^T \in H\} = \frac{b(1-|\alpha\overline{\alpha}|)}{(\alpha-1)(\overline{\alpha}-1)}.$$

We got that if $\pi \in \mathcal{P}$ then $-B_{17} \leq \mathcal{R}(\pi) \leq -A_{17}$ and $-B_{18} \leq \mathcal{I}(\pi) \leq 0$.

4.2.5. Case $\alpha > 0$, $\overline{\alpha} < 0$, $|\alpha| = |\overline{\alpha}|$

In this case $b = -2a$. Since $(a,b) = 1$ and $b > 0$ therefore $a = -1$ and $b = 2$. Clearly, each member of γ_5 is positive and each member of γ_6 is negative. Hence

$$A_{19} := \min\{x \mid [x,y]^T \in H\} = \gamma_6(|N|-1) = -\frac{(|\alpha\overline{\alpha}|-1)^2}{(\alpha^2-1)(\overline{\alpha}^2-1)} = -1, \quad \text{and}$$

$$B_{19} := \max\{x \mid [x,y]^T \in H\} = \gamma_5(|N|-1) = \frac{(b^2E)(|\alpha\overline{\alpha}|-1)}{(\alpha^2-1)(\overline{\alpha}^2-1)} = 1.$$

Since each member of γ_7 is 0 and each member of γ_8 is negative, therefore

$$A_{20} := \min\{y \mid [x,y]^T \in H\} = 0,$$

$$B_{20} := \max\{y \mid [x,y]^T \in H\} = -\gamma_8(|N|-1) = \frac{b(|\alpha\overline{\alpha}|-1)^2}{(\alpha^2-1)(\overline{\alpha}^2-1)} = 2.$$

We have that if $\pi \in \mathcal{P}$ then $-1 \leq \mathcal{R}(\pi) \leq 1$ and $-2 \leq \mathcal{I}(\pi) \leq 0$. By Lemma 5 we have that $0, -\omega, -2\omega \in \mathcal{P}$. Clearly, $-1 = -1 - \alpha\overline{\alpha} + \alpha\overline{\alpha}$ and $\overline{\alpha} = 1 - 2\omega = 0 - \alpha$. Keeping in mind Lemma 1 and 2 it is easy to see that since $\overline{\alpha} \equiv 0 \pmod{\alpha}$, therefore $\Phi(-1-2\omega) = -2\omega$, since $|\alpha\overline{\alpha}| = 4E+1$ and $-2E - \omega \equiv 0 \pmod{\alpha}$, therefore $\Phi(-1-\omega) = \Phi(1-\omega) = -\omega$. Obviously, $\Phi(1) = 0$. Putting everything together we have that $\mathcal{G}(\mathcal{P}) = \{0 \to 0, -\omega \to \to -\omega, -2\omega \to -2\omega, -1 \to \overline{\alpha} \to -1\}$.

Remark. The estimated values $\min\{x \mid [x,y]^T \in H\}, \max\{x \mid [x,y]^T \in \in H\}$, $\min\{y \mid [x,y]^T \in H\}, \max\{y \mid [x,y]^T \in H\}$ are sharp. These values can be used to plot or to compute some topological properties of the set H.

We can formulate the results proved in this section in a theorem.

Theorem. *If $\pi \in \mathcal{P}$ in (α, D) then there are constants $c_i \in \mathbb{R}$ $(i = = 1, \ldots, 4)$ for which $c_1 \leq \mathcal{R}(\pi) \leq c_2$, $c_3 \leq \mathcal{I}(\pi) \leq c_4$ and c_i can explicitly be given.*

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