SOLVED AND OPEN PROBLEMS IN THE REGULARITY THEORY OF FUNCTIONAL EQUATIONS

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Abstract. Although much of the regularity problems concerning the functional equation

$$f(x) = h\Big(x, y, f\big(g_1(x, y)\big), \dots, f\big(g_n(x, y)\big)\Big),$$
$$(x, y) \in D \subset \mathbb{R}^r \times \mathbb{R}^s,$$

with unknown function f — i.e., the general regularity problem of explicit functional equations without iteration — are solved, several open problems remained. We give a survey of results and open problems and we consider connections with Hilbert fifth problem, with regularity of composite and implicite equations and with regularity problem of partial differential equations and variational problems. We consider also algorithmic methods implemented in computer algebra systems.

1. Introduction

From invariance or covariance Cauchy's equation:

$$f(x+y) = f(x)g(y) \qquad x, y \in \mathbb{R}.$$

If there exists an x for which $f(x) = 0$, then $f \equiv 0$, g arbitrary, otherwise $g(y) = f(y)/f(0)$ and $g(x+y) = g(x)g(y)$, i.e., g is multiplicative. Measurable solutions are analytic. An elegant introduction of exponential function.

Sincov equation (for example from introduction of thermodinamical temperature scale):

$$f(x_1, x_2) = f(x_1, y) + f(y, x_2), \qquad x_1, x_2, y \in \mathbb{R}.$$

No regularity, $f(x_1, x_2) = g(x_1) - g(x_2)$.

Basic equation of information (Z. Daróczy):

$$f(x) + f\left(\frac{y}{1-x}\right) = f(y) + f\left(\frac{x}{1-y}\right),$$
$$0 < x, y, x+y < 1.$$

Continuous solutions are in $\mathcal{C}^\infty$. (General method: Andrade 1900, Kac, 1937). Measurability implies continuity: Lee, 1964. Today even with computer can we prove this (S. Czirbusz).

Functional equations come from mathematics, physics, economy, etc.

In his celebrated address to the 1900 International Congress of Mathematicians, in his fifth problem Hilbert asked (in the language of present day mathematics) whether is it true, that every locally Euclidean group is a Lie group? If φ is a homeomorphism mapping $\mathbb{R}^n$ onto a neighbourhood of the unit element e of the group G, such that $\varphi(0) = e$, then we can copy the group operation locally to $\mathbb{R}^n$:

$$(x, y) \mapsto \psi(x, y) = \varphi^{-1}\big(\varphi(x) \cdot \varphi(y)\big).$$

The group axioms give a system of functional equations to ψ, for example associativity became

$$\psi\big(x, \psi(y, z)\big) = \psi\big(\psi(x, y), z\big).$$

In this way, roughly speaking, the problem asks whether we can replace φ with an equivalent homeomorphism so that the corresponding ψ is analytic and satisfies the same system of functional equations. In the second part of his fifth problem Hilbert goes on as follows:

"Moreover, we are thus led to the wide and interesting field of functional equations which have been heretofore investigated usually only under the assumption of the differentiability of the functions involved. In particular the functional equations treated by Abel (Oeuvres, vol. 1, pp. 1, 61, 389) with so much ingenuity ... and other equations occurring in the literature of mathematics, do not directly involve anything which necessitates the requirements of the differentiability of the accompanying functions. ... In all these cases, then, the problem arises: *In how far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption?" (Hilbert's emphasis).*

After this, Hilbert quotes a result of Minkowsky, that under certain conditions, the solutions of the functional inequality

$$f(x+y) \leq f(x) + f(y) \qquad x, y \in \mathbb{R}^n$$

are partially differentiable, and remarks, that certain functional equations, for example the system of functional equations

$$f(x+\alpha) - f(x) = g(x),$$
$$f(x+\beta) - f(x) = 0,$$

where α, β are given real numbers, may have solutions f which are continuous but not differentiable, even if the given function g is analytic.

Regularity for functional equations in a wider sense: problems XIX., XX.: Calculus of variation, elliptic partial differential equations. Connection in both direction: analyticity usually comes from regularity of solutions of differential equations; E. Gselmann's result; Campanato spaces and Morrey lemma.

2. General strategy

A general strategy to solve functional equations:

(A) "Weak" regularity implies "strong" regularity;
(B) Get a differential equation (integral equation, etc.);
(C) Solve this equation.

Today: algorithms for each step (Zs. Páles, A. Gilányi, A. Házy, S. Czirbusz).

Equations with one variable: Hilbert's counterexample.

Iterated equations: Aczél–Benz equation:

$$f\big(x + f(y)\big) = f(x) + f\big(x + y - f(x)\big).$$

Continuous real solutions, Z. Daróczy, 1980. The function $x \mapsto \frac{1}{2}\big(x \pm |x|\big)$ is a solution.

"Most general" equation without iteration:

$$H\Big(x, y, f\big(G(x,y)\big), f_0\big(G_0(x,y)\big), \dots, f_n\big(G_n(x,y)\big)\Big) = 0$$

where the f's are the unknown functions. No regularity phenomenon.

"Explicit form":

$$f(t) = h\Big(t, y, f_0(y), f_1\big(g_1(t, y)\big), \dots, f_n\big(g_n(t, y)\big)\Big).$$

Several regularity theorem states, that if the given functions are "sufficiently nice" and the unknown functions $f_1, f_2, \dots, f_n$ have a regularity property, then the function f has a regularity property which is "one better":

A general strategy to prove regularity:

(I) Measurability implies continuity.

(II) Baire property implies continuity.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p times continuously differentiable solutions are $p+1$ times continuously differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

3. The main problem

Problem. *Let X, Y, and Z be open subsets of $\mathbb{R}^r$, $\mathbb{R}^s$, and $\mathbb{R}^t$, respectively, and let D be an open subset of $X \times Y$ and let W be an open subset of $D \times Z^n$. Let $f : X \to Z$, $g_i : D \to X$ $(i = 1, 2, \dots, n)$, and $h : W \to Z$ be functions. Suppose that*

(P1) *if $(x, y) \in D$ then*

$$\Big(x, y, f\big(g_1(x, y)\big), \dots, f\big(g_n(x, y)\big)\Big) \in W$$

and

$$f(x) = h\Big(x, y, f\big(g_1(x, y)\big), \dots, f\big(g_n(x, y)\big)\Big);$$

(P2) *h is in $\mathcal{C}^\omega$;*

(P3) *g_i is in $\mathcal{C}^\omega$ and for each $x \in X$ there exists a y for which $(x, y) \in D$ and $\dfrac{\partial g_i}{\partial y}(x, y)$ has rank $r = \dim(X)$ $(i = 1, 2, \dots, n)$.*

Is it true that every f, which is in $\mathcal{C}^{-1}$ (i.e., measurable or has the Baire property) is in $\mathcal{C}^\omega$?

Notice the "strong rank condition" that rank $\dfrac{\partial g_i}{\partial y} = r$, which implies $\dim(Y) = s \geq r$.

Variants of the problem. It is possible to substitute the $\mathcal{C}^\omega$ by $\mathcal{C}^\infty$ in the above problem. Clearly, this problem is even more important because most often regular solutions of functional equations are obtained by reduction to differential equations. More generally, we can investigate the "$\mathcal{C}^p$-problem", where $1 \leq p \leq \infty$ or $p = \omega$ ($\omega > \infty$). Since it is not a strong restriction to suppose that the given functions are smoother than we expect the solution to be, it can be useful to investigate the "$\mathcal{C}^p$-$\mathcal{C}^q$ problem": We suppose that the given functions are in $\mathcal{C}^p$, and ask whether the solution f is in $\mathcal{C}^q$? Here $0 \leq q \leq \omega$, $1 \leq p \leq \omega$, and $q \leq p$. For example, in some theorems we assume that the functions g_i are twice continuously differentiable to obtain that f is continuously differentiable, or to prove that f is continuous we will assume that the functions g_i are continuously differentiable. More refined $\mathcal{C}^{p,\alpha}$–$\mathcal{C}^{q,\beta}$ questions can be also considered, where $\mathcal{C}^{p,\alpha}$ means the class of those $\mathcal{C}^p$-functions whose p-th derivative is locally Hölder continuous with exponent α. These refinements are very important because of composite equations. About regularity of solutions of composite equations see M. Laczkovich, Zs. Páles, A. Gilányi.

Partial answer. *If there exists a compact subset C of X such that for each $x \in X$ there exists a $y \in Y$ satisfying $g_i(x, y) \in C$ besides other conditions, then every $\mathcal{C}^{-1}$ solution f is in $\mathcal{C}^{\infty}$.*

Partial answer. *If X is a compact manifold then every $\mathcal{C}^{-1}$ solution f is in $\mathcal{C}^{\infty}$.*

Partial answer. *If X is a locally compact manifold and there exists a compact subset C of X such that for each $x \in X$ there exists a $y \in Y$ satisfying $g_i(x, y) \in C$ besides other conditions, then every $\mathcal{C}^{-1}$ solution f is in $\mathcal{C}^{\infty}$.*

The above problem is well formed. We can prove by simple counterexamples that none of the conditions of the problem above can be omitted without introducing new conditions. This supports our calling this problem "the fundamental regularity problem of non-composite functional equations with several variables". On the other hand, these counterexamples show us the limits of possible changes in the conditions.

The implicit equation

$$f(x)f(y)f(x+y) = 0$$

shows that no regularity is satisfied in general for implicit equations; every function $f : \mathbb{R} \to \mathbb{R}$ is a solution for which $f(x) = 0$ if $x \notin]1,2[$. Indeed, if $x, y \in]1,2[$ then $x + y \in]2,4[$, hence the product is zero. Some of these functions are measurable, but non-continuous, continuous but non-differentiable, etc. Another example connected to the equation characterizing Weierstrass's sigma function.

Cauchy's power equation

$$f(yz) = f(y)f(z)$$

shows that if the equation is implicit in the sense that substituting new variables we cannot convert it to the form $f(x) = \cdots$, then also no regularity is satisfied in general. Indeed, here all functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^c$ and $f(x) = |x|^c \operatorname{sgn} x$ are solutions; $c \in \mathbb{R}$ and 0^c understood to be 0. Some of these functions are measurable but non-continuous, etc.

That the domain D of the functional equation in the above problem has to be open seems to be a technicality: otherwise it would be harder to speak about the differentiability of the inner functions g_i. That it is not possible to omit this condition, even if the functions g_i are very nice, is shown by the example of Cauchy's exponential equation. Substituting a new variable we obtain

$$f(x) = f(x - y)f(y).$$

On any domain D containing the origin $(0, 0)$ and the open set $D^0 = \big\{(x, y) : 0 < y < x\big\}$ but contained in the closure of D^0, the function $f(0) = 1$, $f(x) = 0$ for $x > 0$ is a measurable but non-continuous solution.

If the outer function h in the above problem is not in $\mathcal{C}^p$, then the solution need not be in $\mathcal{C}^p$, $0 \le p \le \omega$. This is proved by the following simple example: Let $H : \mathbb{R} \to \mathbb{R}$ be a one-to-one

Borel function, $f : \mathbb{R} \to \mathbb{R}$, and let us consider Cauchy's functional equation for $H \circ f$, i. e.,

$$H\big(f(x+y)\big) = H\big(f(x)\big) + H\big(f(y)\big), \qquad x, y \in \mathbb{R}.$$

This equation is equivalent to the equation

$$f(x) = h\big(f(y), f(x+y)\big), \qquad x, y \in \mathbb{R},$$

where h is defined by $h(z_1, z_2) = H^{-1}\big(H(z_2) - H(z_1)\big)$ whenever $z_1, z_2 \in \mathbb{R}$. Hence f is a measurable [continuous, etc.] solution of the first equation if and only if f is a measurable [continuous, etc.] solution of the second one. But if f is measurable, then $H \circ f$ is measurable too, hence with some constant c we have $H\big(f(x)\big) = cx$, i. e., $f(x) = H^{-1}(cx)$ for all $x \in \mathbb{R}$. If H^{-1} is measurable but non-continuous [continuous but non-differentiable, etc.] then we see that the second equation has a measurable [continuous, etc.] solution, which is non-continuous [non-differentiable, etc.].

A very similar example shows that if at least one of the inner functions g_i in the problem above is not in $\mathcal{C}^p$, $0 \le p \le \omega$, then the solution f need not be in $\mathcal{C}^p$. Let $G : \mathbb{R} \to \mathbb{R}$ be a one-to-one

Borel function with the property that if $A \subset \mathbb{R}$ is a set having Lebesgue measure zero, then $G^{-1}(A)$ also has Lebesgue measure zero. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let us consider Cauchy's functional equation for $f \circ G$, i. e.,

$$f\big(G(X+Y)\big) = f\big(G(X)\big) + f\big(G(Y)\big), \qquad X, Y \in \mathbb{R}.$$

Substituting $x = G(X)$, $y = G(Y)$, we obtain that this equation is equivalent to the functional equation

$$f(x) = f\big(g(x, y)\big) - f(y), \qquad x, y \in \mathbb{R},$$

where g is defined by $g(x, y) = G\big(G^{-1}(x) + G^{-1}(y)\big)$ whenever $x, y \in \mathbb{R}$.

Let us observe that in the special case $G(X) = X^3$ we obtain that $f(x) = cx^{1/3}$. In this case in equation we have $g(x, y) = (x^{1/3} + y^{1/3})^3$. The function g is continuous on $\mathbb{R} \times \mathbb{R}$ and infinitely many times differentiable at each point (x, y), $x \neq 0 \neq y$. Even the function $y \mapsto g(0, y) = y$ is differentiable with a non-vanishing derivative. But for $x = 0$ there is no such $y \in \mathbb{R}$ for which g is differentiable at (x, y). This shows that we cannot omit the differentiability of the inner function g.

This example also shows that we cannot omit the condition that for each x there is a y such that $(x, y) \in D$.

Finally, that the rank of $\frac{\partial g_i}{\partial y}$ has to be equal to r, is shown by the simple example obtained from Cauchy's power equation by introducing new variables:

$$f(x) = f(y)f(x/y), \qquad x \in \mathbb{R},\ 0 \neq y \in \mathbb{R}.$$

Here, for $x = 0$ there is no y, for which $\frac{\partial g}{\partial y}(0, y) \neq 0$, where $g(x, y) = x/y$. Among real-valued solutions are $f(x) = |x|^c$ and $f(x) = |x|^c \operatorname{sgn} x$, $c \in \mathbb{R}$, where 0^c is understood to be 0. Some of these are measurable but non-continuous, continuous but non-differentiable, etc. The same equation yields a counterexample in higher dimensions, too, if $x, y \in \mathbb{R}^r$, none of the coordinates of y is zero, and the division x/y is understood coordinatewise. Here among the solutions are the functions $f(x) = \prod_{i=1}^r f_i(x_i)$ where each function f_i has the form $f_i(x_i) = |x_i|^{c_i}$ or $f_i(x_i) = |x_i|^{c_i} \operatorname{sgn} x_i$.

Another example that the rank of $\frac{\partial g_i}{\partial y}$ has to be equal to r is Sincov's equation

$$f(x_1, x_2) = f(x_1, y) + f(y, x_2).$$

The general solution $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of this equation is $f(x_1, x_2) = g(x_2) - g(x_1)$, where $g : \mathbb{R} \to \mathbb{R}$ is any function. Indeed, any such function f is a solution. To see that an arbitrary solution f can be represented in this form, let us define $g(x) = -f(x, c)$, where $c \in \mathbb{R}$ is an arbitrary constant.

4. The transfer principle

There are several tricks that make it possible to use our results in cases where seemingly they cannot be used directly: see the chapter containing an application. There is one very important "trick" we have to mention here explicitly. Seemingly, equation (P1) in the problem is rather special in the following respect: it contains only one unknown function. But this restriction is not so strong as it seems to be. If we have a general noncomposite functional equation with several variables and several unknown functions, and we can express *each* unknown function from it, then (after writing different variables in each equation) we may consider a vector-valued function having the different unknown functions as coordinates and use results concerning equations with only one, but vector-valued unknown function. This clearly shows how important it is to discuss vector-valued unknown functions with vector variables.

A simple example. We illustrate this "transfer principle" on the simple example of Pexider's equation

$$f_1(x+y) = f_2(x) + f_3(y), \qquad f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}.$$

From this equation we obtain the following three:

$$\begin{aligned} f_1(x_1) &= f_2(x_1 - y_1) + f_3(y_1), \\ f_2(x_2) &= f_1(x_2 + y_2) - f_3(y_2), \\ f_3(x_3) &= f_1(x_3 + y_3) - f_2(y_3). \end{aligned}$$

Now with the notation $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, for the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x) = f(x_1, x_2, x_3) = \big(f_1(x_1), f_2(x_2), f_3(x_3)\big)$, we obtain the equation

$$f(x) = h\big(f(x_3+y_3, x_1-y_1, y_2), f(x_2+y_2, y_3, y_1)\big),$$

where with the notation $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ the function $h : \mathbb{R}^6 \to \mathbb{R}^3$ is defined by $h(u, v) = (u_2 + v_3, v_1 - u_3, u_1 - v_2)$. Let us observe that this equation satisfies all the conditions of the Problem.

We remark that the condition that *all* unknown functions have to be expressible from the original equation cannot be omitted. The following simple examples show this. By equation

$$f(x) = f(x+y) - f(y) + f_0(x+y)f_0(y)f_0(x+2y),$$
$$f, f_0 : \mathbb{R} \to \mathbb{R}$$

the functions f, f_0, where $f(x) = cx$ and f_0 is any function which is zero outside $]1,2[$ gives a solution. By the equation

$$f(x) = f(x+y) - f(y) + f_0(e^x y^3) - f_0(e^x y^2)f_0(y),$$
$$f, f_0 : \mathbb{R} \to \mathbb{R}$$

the functions f, f_0, where $f(x) = cx$ and $f_0(x) = |x|^{c_0}$ for some $c_0 \in \mathbb{R}$, where 0^{c_0} understood to be zero, yields a solution.

The general transfer principle can be formulated and easily proved.

5. The main results

Theorem. *If h is continuous and the functions g_i are continuously differentiable then every solution $f \in \mathcal{C}^{-1}$ is continuous.*

Theorem. *If the functions h and g_i are p times continuously differentiable, then every almost everywhere differentiable solution f is p times continuously differentiable $(1 \leq p \leq \infty)$.*

Theorem. *If the functions h and g_i are p times continuously differentiable, then every solution f which has locally bounded variation is p times continuously differentiable $(1 \leq p \leq \infty)$.*

Theorem. *If the functions h and g_i are twice continuously differentiable, then every solution f which is locally Hölder continuous for some $0 < \alpha < 1$, is locally Hölder continuous for all $0 < \alpha < 1$.*

Theorem. *If the functions h and g_i are* $\max\{2, p\}$ *times continuously differentiable and there exists a compact subset C of X such that for each $x \in X$ there exists a $y \in Y$ satisfying $g_i(x, y) \in C$ besides other conditions in (P3), then every solution $f \in \mathcal{C}^{-1}$ is p times continuously differentiable* $(1 \le p \le \infty)$.

The additional compactness condition in this theorem is retained by the transfer principle.

Theorem. *If $t = 1$, $n = 2$, $g_1(x, y) \equiv y$, and the function h is p times and the function g_2 is* $\max\{2, p\}$ *times continuously differentiable, then every solution $f \in \mathcal{C}^{-1}$ is p times continuously differentiable* $(1 \le p \le \infty)$.

Theorem. *If equation (P1) has the following special form:*

$$f(x) = \sum_{i=1}^{n} h_i\Big(x, y, f\big(g_i(x, y)\big)\Big),$$

where the functions $h_i : D \times Z \to \mathbb{R}^t$ are p times, the functions g_i are $\max\{2, p\}$ *times continuously differentiable, then every solution $f \in \mathcal{C}^{-1}$ is p times continuously differentiable* $(1 \le p \le \infty)$.

6. A nontrivial example

A problem of W. Sander. The regularity of the functional equation

$$G(x) + (1-x)^{\alpha} F_1\left(\frac{u}{1-x}\right) + (1-x)^{\beta} F_2\left(\frac{u}{1-x}\right)$$
$$= G(u) + (1-u)^{\alpha} F_1\left(\frac{x}{1-u}\right) + (1-u)^{\beta} F_2\left(\frac{x}{1-u}\right)$$

will be studied. Our general regularity theorems cannot be applied directly, because two different unknown functions F_1 and F_2 are present in the equation with the same argument. We can derive new equations for which our theorems may be applied. The same method works for more general functional equations. We may get the following

Theorem. *Let $\alpha \neq \beta$ be fixed real numbers, $F_1, F_2, G :]0,1[\to \mathbb{R}$ functions, and suppose that the functional equation*

$$G(x) + (1-x)^{\alpha} F_1\left(\frac{u}{1-x}\right) + (1-x)^{\beta} F_2\left(\frac{u}{1-x}\right)$$
$$= G(u) + (1-u)^{\alpha} F_1\left(\frac{x}{1-u}\right) + (1-u)^{\beta} F_2\left(\frac{x}{1-u}\right)$$

is satisfied whenever $0 < x < 1$, $0 < u < 1$ *and* $0 < x + u < 1$. *If the functions* F_1, F_2, G *are in* $\mathcal{C}^{-1}$, *then they are in* $\mathcal{C}^{\infty}$.

Proof. The functional equation can be rewritten in the form

$$\begin{aligned} G(u) &= G(x) + (1-x)^{\alpha} F_1\left(\frac{u}{1-x}\right) \\ &\quad + (1-x)^{\beta} F_2\left(\frac{u}{1-x}\right) \\ &\quad - (1-u)^{\alpha} F_1\left(\frac{x}{1-u}\right) - (1-u)^{\beta} F_2\left(\frac{x}{1-u}\right), \end{aligned}$$

which will be useful to prove the regularity of G. We want to derive further equations which will be useful to prove the regularity of F_1 and F_2, respectively.

Let us subtract $G(x)$ from both side of the first equation. With the substitution $u = t(1-x)$ we obtain that

$$\begin{aligned} &(1-x)^{\alpha} F_1(t) + (1-x)^{\beta} F_2(t) \\ &\qquad = G(t(1-x)) + (1-t(1-x))^{\alpha} F_1\left(\frac{x}{1-t(1-x)}\right) \\ &\qquad + (1-t(1-x))^{\beta} F_2\left(\frac{x}{1-t(1-x)}\right) - G(x) \end{aligned}$$

is satisfied whenever $0 < x < 1$ and $0 < t < 1$. To eliminate the term containing F_2 on the left hand side, let us substitute y instead if x; multiplying the original equation with $(1-y)^\beta$ and the substituted equation with $(1-x)^\beta$, the difference is

$$\begin{aligned}
&(1-x)^\beta(1-y)^\alpha F_1(t) - (1-y)^\beta(1-x)^\alpha F_1(t)\\
&\qquad = (1-x)^\beta G(t(1-y))\\
&+ (1-x)^\beta(1-t+ty)^\alpha F_1\left(\frac{y}{1-t(1-y)}\right)\\
&\qquad + (1-x)^\beta(1-t+ty)^\beta F_2\left(\frac{y}{1-t(1-y)}\right)\\
&\qquad - (1-x)^\beta G(y) - (1-y)^\beta G(t(1-x))\\
&- (1-y)^\beta(1-t+tx)^\alpha F_1\left(\frac{x}{1-t(1-x)}\right)\\
&\qquad - (1-y)^\beta(1-t+tx)^\beta F_2\left(\frac{x}{1-t(1-x)}\right)\\
&\qquad + (1-y)^\beta G(x).
\end{aligned}$$

The multiplier $(1-x)^\beta(1-y)^\alpha - (1-y)^\beta(1-x)^\alpha$ of F_1 on the left hand side is not zero if $x \neq y$.

Dividing both side with it we obtain

$$
\begin{aligned}
F_1(t) = \Bigg(& (1-x)^\beta G(t(1-y)) \\
& + (1-x)^\beta (1-t+ty)^\alpha F_1\left(\frac{y}{1-t(1-y)}\right) \\
& + (1-x)^\beta (1-t+ty)^\beta F_2\left(\frac{y}{1-t(1-y)}\right) \\
& - (1-x)^\beta G(y) - (1-y)^\beta G(t(1-x)) \\
& - (1-y)^\beta (1-t+tx)^\alpha F_1\left(\frac{x}{1-t(1-x)}\right) \\
& - (1-y)^\beta (1-t+tx)^\beta F_2\left(\frac{x}{1-t(1-x)}\right) \\
& + (1-y)^\beta G(x) \Bigg) \\
& \cdot \Big((1-x)^\beta (1-y)^\alpha - (1-y)^\beta (1-x)^\alpha \Big)^{-1},
\end{aligned}
$$

whenever $0 < x, y, t < 1$ and $x \neq y$. Similarly we get that

$$
\begin{aligned}
F_2(t) = \Bigg(& (1-x)^\alpha G(t(1-y)) \\
& + (1-x)^\alpha (1-t+ty)^\beta F_2\left(\frac{y}{1-t(1-y)}\right) \\
& + (1-x)^\alpha (1-t+ty)^\alpha F_1\left(\frac{y}{1-t(1-y)}\right) \\
& - (1-x)^\alpha G(y) - (1-y)^\alpha G(t(1-x)) \\
& - (1-y)^\alpha (1-t+tx)^\beta F_2\left(\frac{x}{1-t(1-x)}\right) \\
& - (1-y)^\alpha (1-t+tx)^\alpha F_1\left(\frac{x}{1-t(1-x)}\right) \\
& + (1-y)^\alpha G(x)\Bigg) \\
& \cdot \Big((1-x)^\alpha (1-y)^\beta - (1-y)^\alpha (1-x)^\beta\Big)^{-1},
\end{aligned}
$$

whenever $0 < x, y, t < 1$ and $x \neq y$.

Now we may apply the general regularity theorems.

7. With few variables

The first general regularity results which overcome the difficulty that the“stronk rank condition” is not satisfied were given by Światak. She applied her distribution method to generalizations of the mean value equation. She investigated the equation

$$\sum_{i=1}^{n} h_i(x,y) f\big(g_i(x,y)\big) = h_0(x,y),$$
$$x \in \mathbb{R}^r,\ y \in Y \subset \mathbb{R}^s.$$

with unknown function f and proved that continuous solutions are in $\mathcal{C}^\infty$.

We may consider "measurability implies continuity" type result for the general explicit nonlinear functional equation (P1) without the strong rank condition from (P3) to the inner functions. In the spirit of the "bootstrap" method corresponding to (I)–(VI) we introduce a sequence of properties, which — roughly speaking — are between measurability and continuity. This sequence of properties gives a starway to come down from measurability to continuity. Similar principles apply to Baire property and — in the quasilinear case — continuity. The general theory is treated in 3 papers, $\approx$70 pages.

Corollary. *Let $X \subset \mathbb{R}^r$ be an open set and $f : X \to \mathbb{R}^m$ a function. Suppose that*

(LFES) the linear functional equations

$$f(x) = h_{i,0}(x,y) + \sum_{j=1}^{n_i} h_{i,j}(x,y) f\big(g_{i,j}(x,y)\big)$$

are satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set), moreover

(S2) $D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, the functions $h_{i,0} : D_i \to \mathbb{R}^m$ and $h_{i,j} : D_i \to \mathbb{R}$ are in $\mathcal{C}^1$, the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^2$, moreover

(D) for each $x \in X$ and for each proper linear subspace V of $\mathbb{R}^r$ there exists an $i \in I$ and a y such that $(x,y) \in D_i$ and

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{im}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V)$$

whenever $1 \le j \le n_i$.

Then $f \in \mathcal{C}^0$ implies $f \in \mathcal{C}^1$.

Observe, that if $\dim(Y_i) > 0$, then the dimension condition (D) is satisfied “in general”, because "in general"

$$\begin{aligned}
&\det\left(\frac{\partial g_{i,j}}{\partial x}(x,y)\right) \neq 0,\\
&\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V)\right) = \dim(V),\\
&\operatorname{rank}\left(\frac{\partial g_{i,j}}{\partial y}(x,y)\right) = \min\big\{r, \dim(Y_i)\big\} > 0.
\end{aligned}$$

Corollary. *Let $X \subset \mathbb{R}^r$ be an open set and $f : X \to \mathbb{R}^m$ a function. Suppose that*

(FES) *we have*

$$\Big(x, y, f\big(g_{i,1}(x,y)\big), \dots, f\big(g_{i,n_i}(x,y)\big)\Big) \in W_i$$

and the functional equation

$$f(x) = h_i\Big(x, y, f\big(g_{i,1}(x,y)\big), \dots, f\big(g_{i,n_i}(x,y)\big)\Big)$$

is satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set), moreover

(SI) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, W_i is an open subset of $D_i \times (\mathbb{R}^m)^{n_i}$, all the partial derivatives $\partial_t^{\alpha_0} \partial_{z_1}^{\alpha_1} \dots \partial_{z_{n_i}}^{\alpha_{n_i}} h_i$ of the functions $h_i : W_i \to \mathbb{R}^m$ are continuously differentiable, the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^\infty$, moreover*

(D) *for each $x \in X$ and for each proper linear subspace V of $\mathbb{R}^r$ there exists an $i \in I$ and a y such that $(x,y) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{im}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V)$$

whenever $1 \le j \le n_i$.

Then $f \in \mathcal{C}^1$ implies $f \in \mathcal{C}^\infty$.

Critical subspaces. Let $X \subset \mathbb{R}^r$ be an open set and for each $i \in I$ let $D_i \subset X \times Y_i$ be an open set, where Y_i is a Euclidean space and let the functions $g_{i,j} : D_i \to X$, $1 \le j \le n_i$ be in $\mathcal{C}^1$. Suppose, that for each $x \in X$ there is a $i \in I$ and a y such that $(x, y) \in D$. For a proper linear subspace V of $\mathbb{R}^r$ we will say that it is a critical subspace at x if for each $i \in I$ and for each y for which $(x, y) \in D_i$ we have

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{im}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) \le \dim(V)$$

for some $1 \le j \le n_i$.

It is clear, that the dimension condition (D) can be formulated on the way that there is no critical subspace for any $x \in X$. It is also clear that if a linear subspace V is critical then any proper linear subspace of $\mathbb{R}^r$ containing V is critical too. Hence it is enough to consider minimal critical linear subspaces.

An example. Let us consider equation

$$f(x+t,y)+f(x,y+t^2)=2f(x,y), \quad x,y,t\in\mathbb{R}.$$

For this equation a proper linear subspace V is critical if and only if

$$\dim\Big(\mathbb{I}(V)+\big\{(u,0):u\in\mathbb{R}\big\}\Big)\le 1,$$

i. e. $V=V_1=\big\{(u,0):u\in\mathbb{R}\big\}$, or

$$\dim\Big(\mathbb{I}(V)+\big\{(2tu,0):u\in\mathbb{R}\big\}\Big)\le 1$$

for all $t\in\mathbb{R}$, i. e. $V=V_2=\big\{(0,u):u\in\mathbb{R}\big\}$.

Substituting $x+t$ with x, for the new equation

$$f(x,y)=2f(x-t,y)-f(x-t,y+t^2), \quad x,y,t\in\mathbb{R}$$

a proper linear subspace V is critical if and only if

$$\dim\Big(\mathbb{I}(V)+\big\{(-u,2tu):u\in\mathbb{R}\big\}\Big)\le 1$$

for all $t\in\mathbb{R}$ or

$$\dim\Big(\mathbb{I}(V)+\big\{(-u,0):u\in\mathbb{R}\big\}\Big)\le 1.$$

Hence for these two equations only V_2 remains critical.

Substituting $x + t^2$ with x, for the new equation

$$f(x,y) = 2f(x,y-t^2)-f(x+t,y-t^2), \quad x,y,t \in \mathbb{R}$$

a proper linear subspace V is critical if and only if

$$\dim\Big(\mathbb{I}(V) + \big\{(-u,-2tu) : u \in \mathbb{R}\big\}\Big) \leq 1$$

or

$$\dim\Big(\mathbb{I}(V) + \big\{(0,-2tu) : u \in \mathbb{R}\big\}\Big) \leq 1$$

for all $t \in \mathbb{R}$. Hence for all three equations no critical subspace remains.

Sincov equation. Sincov equation

$$f(x_1, x_2) = f(x_1, y) + f(y, x_2), \qquad x_1, x_2, y \in \mathbb{R}$$

does not belong to the class of generalized mean value equations, but if we try to apply our general results, we find the critical subspaces V_1 and V_2 as above, but in this case we cannot remove these with substitutions. Let us observe the connection of the critical directions with the general solution $f(x_1, x_2) = g(x_1) - g(x_2)$.

Some examples. In her paper H. Światak illutrate her regularity results with the following examples:

$$f(x+t,y)+f(x,y+t^2)=2f(x,y),$$

$$f(x+t,y)+f(x,y+t^2)+f(x,y-t^2)=3f(x,y),$$

$$f(x+t,y)+t^2f(x,y+t)-(1+t^2)f(x,y)=t(2x+t),$$

$$f(x+t,y)+t^2f(x,y+t)+(t-t^2-1)f(x,y)$$
$$=t(x^2+2x+t),$$

$$f(x+t,y)+t^2f(x,y+t)+(t-t^2-1)f(x,y)$$
$$=t(x^2+2x+t+t^3+2yt+y^2),$$

$$f(x+t,y)+t^2f(x,y+t)-(1+t^2)f(x,y)$$
$$=t(2x+t+t^3+2yt^2),$$

$$f(x+t,y)+f(x-t,y)+f(x,y+t^2)+f(x,y-t^2)$$
$$=4f(x,y),$$

$$f(x+t,y)+f(x-t,y)+f(x,y+t^3)+f(x,y-t^3)$$
$$=4f(x,y),$$

where the unknown function is $f:\mathbb{R}^2\to\mathbb{R}$.

All these equations can be easily handled using the corollaries similarly as in the above example.

Corollary. *Let $X \subset \mathbb{R}^r$ be an open set, $f : X \to \mathbb{R}^m$ a function and let $K \subset \{0, 1, \ldots, r\}$ containing 0 and r. Suppose that*

(FES) *we have*

$$\Big(x, y, f\big(g_{i,1}(x,y)\big), \ldots, f\big(g_{i,n_i}(x,y)\big)\Big) \in W_i$$

and the functional equation

$$f(x) = h_i\Big(x, y, f\big(g_{i,1}(x,y)\big), \ldots, f\big(g_{i,n_i}(x,y)\big)\Big)$$

is satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set), moreover

(S1) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, W_i is an open subset of $D_i \times (\mathbb{R}^m)^{n_i}$, for all $y \in Y_i$, h_i is continuous in the other variables and the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^1$, moreover*

(CD) *for each $x_0 \in X$ and for each proper linear subspace V_0 of $\mathbb{R}^r$ with $k_0 = \dim(V_0) \in K$ there exists an $i \in I$ and a y_0 such that $(x_0, y_0) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{im}\frac{\partial g_{i,j}}{\partial y}(x,y)\right)$$

is the same constant k in K and greater than k_0 for $1 \leq j \leq n_i$, whenever x is close enough to x_0, y is close enough to y_0 and V is close enough to V_0 in the Grassmann manifold $\mathbb{G}(r, k)$.

Then $f \in \mathcal{C}^{-1}$ implies $f \in \mathcal{C}^0$.

Here again if $\dim(Y_i) > 0$, the “constant dimension” condition (CD) is satisfied “in general”, but not if there is a critical subspace for some $x \in X$.

A simple reduction process. In some cases, there is a possibility to reduce the regularity problem of functional equations with "few" variables to the regularity problem of functional equations with "enough" variables. The connection of this method with the general theory were investigated by I. Kovácsvölgyi.

8. Implicit equations

Unnumerable examples. An important type:

$$f_1(x_1)f_2(x_2)\cdots f_m(x_m) = g_1(y_1)g_2(y_2)\cdots g_m(y_m)$$

satisfied almost everywhere with complex valued unknown functions f_i and g_j and given vector valued **funcions** y_j of $(x_1, x_2, \dots, x_m)$. Several such equations come from probability theory. Special cases by K. Lajkó and F. Mészáros.

There is a general “0–1 principle”.

References

1. János Aczél, *Some unsolved problems in the theory of functional equations II*, Aequationes Math. **26** (1984), 255–260.
2. János Aczél, *The state of the second part of Hilbert's fifth problem*, Bull. Amer. Math. Soc. (N.S.) **20** (1989), 153–163.
3. Karl-Groswin Grosse-Erdmann, *Regularity properties of functional equations and inequalities*, Aequationes Math. **37** (1989), 233–251.
4. Herbert Federer, *Geometric measure theory*, Springer, Berlin–Heidelberg–New York, 1969.
3. Nicolas Bourbaki, *Elements of mathematics. General topology*, Addison-Wesley, Reading, Mass.–Palo Alto–London–Don Mills, Ont., 1966.
4. David Hilbert, *Gesammelte Abhandlungen Band III*, Springer Verlag, Berlin–Heidelberg–New York, 1970.
6. Járai Antal, *Invariant extension of Haar measure*, Diss. Math. **233** (1984), 1–26.
5. Antal Járai, *On regular solutions of functional equations*, Aequationes Math. **30** (1986), 21–54.
6. Antal Járai, *Differentiation of parametric integrals and regularity of functional equations*, Grazer Math. Ber. **315** (1991), 45–50.
7. Antal Járai, *On Lipschitz property of continuous solutions of functional equations*, Aequationes Math. **47** (1994), 69–78.
9. Járai Antal, *A Steinhaus type theorem*, Publ. Math. Debrecen **47** (1995), 1–13.
8. Antal Járai, *Regularity properties of functional equa-*

tions. Leaflets in Mathematics, Janus Pannonius University, Pécs, 1996, pp. 1–77.

9. Antal Járai, *Regularity properties of functional equations in several variables*, To appear by Kluwer, pp. 1–363.
10. Antal Járai, *Solutions of functional equations of bounded variations*, Aequationes Math. **61** (2001), 205–211.
11. Antal Járai, *Regularity properties of functional equations on manifolds*, Aequationes Math. **64** (2002), 248–262.
12. Járai Antal, *Solutions of functional equations of bounded variations*, In print, Aequationes Math.
12. Antal Járai, *Baire property implies continuity for solutions of functional equations — even with few variables*, Acta Sci. Math. Szeged **66** (2000), 579–601.
13. Antal Járai, *Measurability implies continuity for solutions of functional equations — even with few variables*, Aequationes Math. **65** (2003), 236–266.
15. Járai Antal, *Continuity implies differentiability for solutions of functional equations — even with few variables* (to appear).
14. Antal Járai, Wolfgang Sander, *A regularity theorem in information theory*, Publ. Math. Debrecen **50** (1997), 339–357.
15. Antal Járai, László Székelyhidi, *Regularization and General Methods in the Theory of Functional Equations. Survey paper*, Aequationes Math. **52** (1996), 10–29.
16. Járai A., *Regularity properties of functional equations in several variables*, Springer Verlag, 2005, pp. 1–363.
19. Michael A. McKiernan, *Boundedness on a Set of Positive Measure and the Mean Value Property Character-*

izes Polynomials an a Space V^n, Aequationes Math. **4** (1970), 31–36.
20. Frank Morgan, *Geometric measure theory. A beginner's guide*, Academic Press, 1988.
16. John C. Oxtoby, *Mass und Kategorie*, Springer, Berlin–Heidelberg–New York, 1971.
17. Halina Światak, *The regularity of the locally integrable and continuous solutions of nonlinear functional equations*, Trans. Amer. Math. Soc. **221(1)** (1976), 97–118.
23. Rolf Trautner, *A covering principle in real analysis*, Quart. J. Math. Oxford **38(2)** (1987), 127–130.

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