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ABSTRACT. Baire category analogues of ,,measurability implies continuity for solutions of a functional equation satisfied almost everywhere" will be studied.

Measurable almost solutions of functional equations investigated frequently. Here a Baire category analogue of such problems is studied.

Notation and terminology. We follow the terminology of [5]. Note that if E is a subset of a topological space X then we say that E is of second category at a point $x \in X$, if $E \cap V$ is of second category for each neighbourhood V of x. Otherwise we say that E is of first category at x or that x is in essence an inner point of E. The set $D(E)$ of all points x for which E is of second category at x is closed, $E \setminus D(E)$ is of first category and $D(E)$ is empty if and only if E is of first category. This is a considerable simpler situation as in the case of measurability, where the following definition and theorem where used (see [6]).

Definition 1. A point $x \in \mathbb{R}^n$ is called an *almost inner point* of a set $A \subset \mathbb{R}^n$, if there is an open ball $\mathbb{U}_r(x)$ centered at x and having radius $r > 0$ such that $\mathbb{U}_r(x) \setminus A$ has Lebesque (outer) measure λ^n equal to zero.

Theorem 1. *For any set $A \subset \mathbb{R}^n$ the set V of almost inner points of A is an open set. This is the maximal open set for which $V \setminus A$ has measure zero.*

For comparision we cite here from [6] the main tool in the measurability case.

An almost Steinhaus-type theorem. *Let X be an r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$. Suppose that $r_i \geq 1$ $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ be a continuously differentiable function. For each $x \in U$ let N_x denote the null space of $F'(x)$. Let A_i be a subset of X_i $(i = 1, \dots, n)$ and suppose that A_i is λ^{r_i} measurable for $2 \leq i \leq n$. Let $a \in U$, $\dim(N_a) = r - m$. Let p_i denote the orthogonal projection of X onto X_i. Suppose that $p_i(N_a) = X_i$ and A_i has density 1 in the point $p_i(a)$ whenever $1 \leq i \leq n$, and let N be an r-dimensional null set. Then $F(a)$ is an almost inner point of $F\big((A_1 \times \cdots \times A_n) \setminus N\big)$.*

The category analog of this theorem reads as follows.

Main tool: an in essence Piccard-type theorem. *Let X be an r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$. Suppose that $r_i \geq 1$ $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ be a continuously differentiable function. For each $x \in U$ let N_x denote the null space of $F'(x)$. Let A_i be a Baire subset of X_i $(i = 2, 3, \dots, n)$. Let $a \in U$, $\dim(N_a) = r - m$. Let p_i denote the orthogonal projection of X onto X_i. Suppose that $p_i(N_a) = X_i$ and A_1 has of second category in the point $p_1(a)$, moreover $p_i(a)$ is in essence an inner point of A_i whenever $2 \leq i \leq n$, and let N be an r-dimensional set of first category. Then $F(a)$ in essence is an inner point of $F\big((A_1 \times \cdots \times A_n) \setminus N\big)$.*

The proof is similar to the proof of Theorem 4.4 in my book [5]. Unfortunately, it does not seems possible to deduce the present theorem from Theorem 4.4 in [5]. Hence we will give a complete proof using some results there.

Proof. Let $k = r - m$. Since $x \mapsto \operatorname{rank} F'(x)$ is lower semicontinuous, and $\operatorname{rank}\big(F'(a)\big) = m$, we may suppose that $\operatorname{rank}\big(F'(x)\big) = m$ whenever $x \in U$. Similarly, choosing a smaller U if necessary, we may suppose that $p_i(N_x) = X_i$ whenever $x \in U$ and $1 \leq i \leq n$; to prove this, suppose to the contrary that there exists an i and for each natural number j there exists an $x_j \in U$ and there exist orthonormal vectors $e_1^j, \ldots, e_{k-r_i+1}^j$ in N_{x_j} such that $x_j \to a$ and $p_i(e_s^j) = 0$ whenever $j = 1, 2, \ldots$ and $1 \leq s \leq k - r_i + 1$. Using the compactness of the unit sphere we can pass to a subsequence and suppose that $e_s^j \to e_s$ if $j \to \infty$. But this proves that the vectors e_s are orthonormal in N_a and $p_i(e_s) = 0$ whenever $1 \leq s \leq k - r_i + 1$, which is a contradiction.

Now, choosing a smaller U if necessary and using the rank theorem, we have that there exist mappings u, p, and v and an open neighborhood V of $b = F(a)$ in $\mathbb{R}^m$ with the following properties: u

maps U onto the open cube I^r, where $I =]-1, 1[$, u is invertible and moreover u and u^{-1} are continuously differentiable; v maps I^m onto V, v is invertible and moreover v and v^{-1} are continuously differentiable; p is the projection

$$p : (x_1, \dots , x_r) \mapsto (x_1, \dots , x_m)$$

of I^r onto I^m; and finally $F = v \circ p \circ u$. We may write I^r as $I^r = T \times Y$ where $T = I^m$ and $Y = I^k$. Let $u(a) = (t_0, y_0) \in T \times Y$. Now let us use some facts from differential geometry (see the book [1] of Dieudonné, mainly 16.8.8).

$U \cap F^{-1}\big(v(t)\big)$ is a closed submanifold of U whenever $t \in T$. The tangent space of this submanifold in a point $x \in U \cap F^{-1}\big(v(t)\big)$ is equal to the subspace N_x of X. Clearly u^{-1} is a diffeomorphism of the closed submanifold $\{t\} \times Y$ of $T \times Y$ onto $U \cap F^{-1}\big(v(t)\big)$. Let $g_i = p_i \circ u^{-1}$ if $1 \le i \le n$. By the choice of U, p_i is a submersion of $U \cap F^{-1}\big(v(t)\big)$ into X_i. Hence the mapping $g_{i,t} : Y \to X_i$ is a submersion too, that is, its derivative has rank r_i whenever $y \in Y$ and $t \in T$.

Now, by Lemma 4.3 from [5], there exist open sets T^* and Y^* such that $t_0 \in T^* \subset T$, $y_0 \in Y^* \subset Y$, and $g_{i,t}(B)$ is of second category whenever $B \subset Y^*$ has second category and $t \in T^*$. Let $X_i^* = X_i$, $A_i^* = A_i$, and g_i^* be the restriction of g_i to $T^* \times Y^*$. Applying Theorem 4.1 and Remark 4.2 from [5] to the sets marked by star we have that the set V of points t for which

$$\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A_i)$$

is of second category is open in T^*. Since g_{i,t_0}^* maps Y^* onto an open neighborhood of $p_i(a)$ and $g^*(t_0, y_0) = p_i(a)$, there exists an open neighborhood W of y_0 in Y^* such that $W \setminus g_{i,t_0}^{*-1}(A_i)$ is of first category if $i = 2, 3, \dots, n$. As in the proof of Remark 4.2 in [5] we obtain that $W \cap g_{1,t_0}^{*-1}(A_1)$ is of second category. This proves that $t_0 \in V$. Clearly $v(V)$ is an open neighborhood of b in $\mathbb{R}^m$. Because u is a diffeomorphism, $u(N)$ is a set of first category, hence $u(N) \cap (T \times Y)$ is a set of first category, too. By the Kuratowsky–Ulam theorem, there exists a first category set $M \subset T$ such that if $t \in T \setminus M$, then $N_t = \{y : (t, y) \in u(N)\}$ is a first category subset of Y. Of course, $v(M)$ is a first category subset of $\mathbb{R}^m$. If $z \in v(V) \setminus v(M)$, then $t := v^{-1}(z) \in V \setminus M$ and the set

$$\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A_i) \setminus N_t \tag{1}$$

has the same category as the set

$$\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A_i)$$

and hence is nonvoid. If y is an element of the set (1), then $F\big(u^{-1}(t,y)\big) = v\big(p(t,y)\big) = v(t) = z$, $u^{-1}(t,y) \notin N$ and $x_i = p_i\big(u^{-1}(t,y)\big) = g_{i,t}^{*}(y) \in A_i$ if $1 \le i \le n$. This means that $F(x_1, \ldots, x_n) = z$.

Corollary 1. *Let U be an open subset of $\mathbb{R}^r \times \mathbb{R}^r$, let N be a first category set of $\mathbb{R}^r \times \mathbb{R}^r$ and $F : (x, y) \mapsto F(x, y)$ a continuously differentiable mapping of U into $\mathbb{R}^r$. Let $B \subset \mathbb{R}^r$ be a Baire set. If $(a, b) \in U$,*

$$\det \frac{\partial F}{\partial x}(a, b) \neq 0, \qquad \det \frac{\partial F}{\partial y}(a, b) \neq 0,$$

$A \subset \mathbb{R}^r$ has second category at the point a and b is in essence an inner point of B, then $F(a, b)$ in essence is an inner point of $F\big((A \times B) \setminus N\big)$.

Corollary 2. *Another form of the previous theorem is the following: Let* A_i *be a Baire subset of* X_i $(i = 2, 3, \dots, n)$*. Let* $a \in U$*. If the rank of the matrix*

$$\frac{\partial F}{\partial(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}(a)$$

is equal to m *for any* $i = 1, 2, \dots, n$ *and* A_1 *has of second category in the point* $p_1(a)$ *moreover* $p_i(a)$ *is in essence an inner point of* A_i *whenever* $2 \leq i \leq n$*, then* $F(a)$ *in essence is an inner point of* $F\big((A_1 \times \dots \times A_n) \setminus N\big)$*.*

Proof. We have to prove that N_a is in general position, i.e., $\dim N_a = r - m$ and $p_i(N_a) = X_i$ $(i = 1, 2, \dots, n)$. But this follows from lemma 23.5 in [5].

Corollary 3. *The previous theorem may be stated in the following global form: If* N_x *is in general position, i.e.,* $\dim N_x = r - m$ *and* $p_i(N_x) = X_i$ *for all* $x \in U$ $(i = 1, 2, \dots, n)$, *moreover* $A_1 \times A_2 \times \cdots \times A_n \subset U$, *the set* A_i *is Baire set* $(i = 2, 3 \dots, n)$ *and is of the second category* $(i = 1, 2, 3, \dots, n)$, *then for any* r*-dimensional set* N *which is of the first category we have that* $F\big((A_1 \times \cdots \times A_n) \setminus N\big)$ *contains a Baire set of second category.*

Corollary 4. *Another global form of the previous theorem is the following: If the rank of the matrix*

$$\frac{\partial F}{\partial(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}(x)$$

is equal to m *for any* $x \in U$ *and for any* $i = 1, 2, \dots, n$*, moreover* $A_1 \times A_2 \times \cdots \times A_n \subset U$*, the set* A_i *is Baire set* $(i = 2, 3, \dots, n)$ *and is of second category* $(i = 1, 2, \dots, n)$*, then for any* r*-dimensional set* N *which is of the first category we have that* $F\big((A_1 \times \cdots \times A_n) \setminus N\big)$ *contains a Baire set of second category.*

Proof. We have to prove that N_x is in general position, i.e., $\dim N_x = r - m$ and $p_i(N_x) = X_i$ for all $x \in U$ $(i = 1, 2, \dots, n)$. But this follows from lemma 23.5 in [5].

We want to illustrate with an example how to apply this theorem and its corollaries to multiplicative type functional equations satisfied except a set of first category to prove that their solutions are equal to nowhere zero function in $\mathcal{C}^\infty$ except a set of first category. The example is not very simple, because we want to illustrate the power of our method.

An example. In their paper [3] Dan Geiger and David Heckerman provide a new characterization of the Dirichlet distribution. Their results show that in certain statistical problems we do not need to suppose that the random variables have Dirichlet distribution, this is automatically implied by the more natural independence conditions. Their method consists of proving that probability considerations result in the functional equation

$$(FE) \qquad f_0(y)\prod_{j=1}^{n} g_j(z_j) = g_0(x)\prod_{i=1}^{k} f_i(w_i).$$

This equation should be satisfied whenever

$$y \in \Delta_{n-1} \quad \text{and} \quad z_j \in \Delta_{k-1} \qquad \text{for } 1 \le j \le n$$

(here Δ_m denotes the m-dimensional open unit simplex in $\mathbb{R}^m$). Furthermore, $x \in \Delta_{k-1}$ and $w_i \in \Delta_{n-1}$ are given functions of y and z_j, $1 \le i \le n$. The functional equation may be considered in a symmetrical way where the x's and w's are the free variables, i. e., (FE) is satisfied for all $x \in \Delta_{k-1}$ and $w_j \in \Delta_{n-1}$, $1 \le j \le n$. In short, there is a complete symmetry with respect to $f \leftrightarrow g$, $x \leftrightarrow y$, $z \leftrightarrow w$, $k \leftrightarrow n$, $i \leftrightarrow j$.

Geiger and Heckerman solve functional equation (FE) by reducing it to a differential equation. To do this they have to suppose that the unknown functions in (FE) are smooth and positive everywhere but we only know that they are density functions of some random variables and hence the functional equation is only satisfied almost everywhere. This leads to the following problem:

Problem. *Suppose that the unknown functions in the functional equation (FE) are density functions — i. e., nonnegative and Lebesgue integrable with integral 1 — and (FE) is satisfied almost everywhere. Does it follow that they are almost equal to positive $\mathcal{C}^\infty$-functions satisfying (FE) everywhere?*

We gave an affirmative answer to this question. We summarise the *only* used key observations about (FE) in the proof of the next theorem.

Key observations.

(1) Both sides are a product of values of unknown functions;

(2) the domains of the unknown functions f_i, $0 \le i \le k$ and g_j, $0 \le j \le n$ are connected open sets;

(3) the mapping

$$F : (y, z_1, \dots, z_n) \mapsto (x, w_1, \dots, w_k)$$

is a one-to-one $\mathcal{C}^1$-mapping of the Cartesian product of the domains of $f_0, g_1, \dots, g_n$ to the Cartesian product of the domains of $g_0, f_1, \dots, f_k$ and considering its inverse mapping $G : (x, w_1, \dots, w_k) \mapsto (y, z_1, \dots, z_n)$ it is also $\mathcal{C}^1$-mapping, hence both are $\mathcal{C}^1$-diffeomorphisms;

(4) fixing any of the vector variables $y, z_1, \dots, z_n$ and choosing any of the dependent vectors $x, w_1, \dots, w_k$, the corresponding mapping is a submersion, i.e., its derivative has rank equal to the dimension of the chosen dependent vector, and the same remains true if we change the role of $y, z_1, \dots, z_n$ and $x, w_1, \dots, w_k$.

Theorem 2. *Let* f_i, $0 \leq i \leq n$ *and* g_j, $0 \leq j \leq k$ *be complex valued measurable functions satisfying (FE) almost everywhere, where the inner functions satisfies (1)–(4) of the key observation. Then either on both sides one of the functions is zero almost everywhere or all of the functions are almost everywhere nonzero.*

Remarks about the proof. (1) We may observe that the proof not use the fact that the functions f_i and g_j are measurable, only that the sets $\{f_i = 0\}$ and $\{g_j = 0\}$ are measurable.

(2) Let us observe that it is not important that multiplication between the unknown functions is commutative and associative, only that there are no zero divisors: the product is zero if and only if one of the terms is zero, otherwise it is nonzero.

(3) Moreover, if one side (or both) of (FE) was multiplied by an arbitrary function $h(y, z_1, \dots, z_n)$ that is everywhere nonzero, the statement would still remain true.

(4) Even much more generally, the "multiplication" on both sides could be depend on the variables $y, z_1, \dots, z_n$ and need not be obtained from binary operations.

(5) The situation can be strongly simplified if all nonzero values of the unknown functions are contracted to be a fixed nonzero value, say 1, and the same is applied to the result of the (maybe variable-dependent) "multiplication". (For example, in the complex valued case, the absolute value of the signum of the unknown functions are taken.)

Using these ideas, the proof of the previous theorem (but with different notation) and the key observations above we obtain the following general theorem:

Zero–one–law for multiplicative functional equations, measure case. *Suppose that the 0–1 valued functions* f_i, $i = 1, \ldots, m$ *and* g_j, $j = 1, \ldots, n$ *are Lebesgue measurable and the functional equation*

$$(MFE) \qquad \prod_{i=1}^{m} f_i(x_i) = \prod_{j=1}^{n} g_j(y_j)$$

is satisfied almost everywhere on $X_1 \times \cdots \times X_m$, *where*

(1) *the domains* X_i *and* Y_j *of the unknown functions* f_i, $i = 1, \ldots, m$ *and* g_j, $j = 1, \ldots, n$, *respectively, are nonvoid connected open sets of finite dimensional Euclidean spaces;*

(2) *the inner functions* $y_1, \ldots, y_n$ *of* $x_1, \ldots, x_m$ *are such that the mapping* $F : (x_1, \ldots, x_m) \mapsto (y_1, \ldots, y_n)$ *is a one-to-one* $\mathcal{C}^1$*-mapping of the Cartesian product of the* X_i*'s to the Cartesian product of the* Y_j*'s and its inverse mapping* $G : (y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_m)$ *is also* $\mathcal{C}^1$*-mapping, hence both are* $\mathcal{C}^1$*-diffeomorphizms;*

(3) *fixing any of the vector variables* x_i $i = 1, \dots, m$ *and choosing any of the dependent vectors* y_j $j = 1, \dots, n$, *the corresponding mapping is a submersion, i. e., its derivative has rank equal to the dimension of the chosen dependent vector, and the same remains true if we change the roles of* x_i, $i = 1, \dots, m$ and y_j $j = 1, \dots, n$.

Then either on both sides one of the functions is zero almost everywhere or all of the functions f_i, $i = 1, \dots, m$ *and* g_j, $j = 1, \dots, n$ *are equal to 1 almost everywhere.*

We are interested here in the category analoge of the above zero–one–law.

Zero–one–law for multiplicative functional equations, category case. *Suppose that the 0–1 valued functions f_i, $i = 1, \dots, m$ and g_j, $j = 1, \dots, n$ have Baire property and the functional equation*

$$(MFE) \qquad \prod_{i=1}^{m} f_i(x_i) = \prod_{j=1}^{n} g_j(y_j)$$

is satisfied except for a set of first category on $X_1 \times \cdots \times X_m$, where

(1) *the domains X_i and Y_j of the unknown functions f_i, $i = 1, \dots, m$ and g_j, $j = 1, \dots, n$, respectively, are nonvoid connected open sets of finite dimensional Euclidean spaces;*

(2) *the inner functions $y_1, \dots, y_n$ of $x_1, \dots, x_m$ are such that the mapping $F : (x_1, \dots, x_m) \mapsto (y_1, \dots, y_n)$ is a one-to-one $\mathcal{C}^1$-mapping of the Cartesian product of the X_i's to the Cartesian product of the Y_j's and its inverse mapping $G : (y_1, \dots, y_n) \mapsto (x_1, \dots, x_m)$ is also $\mathcal{C}^1$-mapping, hence both are $\mathcal{C}^1$-diffeomorphizms;*

(3) *fixing any of the vector variables* x_i $i = 1, \dots, m$ *and choosing any of the dependent vectors* y_j $j = 1, \dots, n$, *the corresponding mapping is a submersion, i. e., its derivative has rank equal to the dimension of the chosen dependent vector, and the same remains true if we change the roles of* x_i, $i = 1, \dots, m$ and y_j $j = 1, \dots, n$.

Then either on both sides one of the functions is zero except a set of first category or all of the functions f_i, $i = 1, \dots, m$ and g_j, $j = 1, \dots, n$ *are equal to 1 almost everywhere.*

Proof. We have to prove that if on one side non of the functions is zero except a set of the first category then all the functions are nonzero except a set of first category. We will use the notation

$$\{f_1 = 0\} = \{x_1 \in \mathrm{dmn}(f_1) : f_1(x_1) = 0\}$$

and the analogous notation $\{f_i = 0\}$, $\{f_i \neq 0\}$, $\{g_j = 0\}$, $\{g_j \neq 0\}$.

First let us suppose that all the sets $\{f_i \neq 0\}$ is of the second category. We prove that if one of the sets $\{f_i \neq 0\}$, $1 \leq i \leq m$ has second category at a point, then this point is in essence an inner point of it. Suppose that at $x_i^0 \in \mathrm{dmn}(f_i)$ the set $\{f_i \neq 0\}$ has second category. For each $k \neq i$ we choose a point $x_k^0 \in \mathrm{dmn}(f_k)$ which is in essence an inner point of $\{f_k \neq 0\}$; this is possible because $\{f_k \neq 0\}$ is a Baire set, hence can be written as the symmetric difference of an open set and a set of first category. Let $(y_1^0, y_2^0, \dots, y_n^0) = F(x_1^0, x_2^0, \dots, x_m^0)$. By our main tool, more exacly by its Proprosition 2, we obtain that y_j^0 is in essence an inner point of $\mathrm{dmn}(g_j)$, and hence $\{g_j \neq 0\}$ has of the second category for $1 \leq j \leq n$.

Now we will consider the functional equation (FE) with free variables y_j, and apply our main tool to the mapping $(y_1, y_2, \dots, y_n) \mapsto x_i$ at the point $(y_1^0, y_2^0, \dots, y_n^0)$. Let us choose for each y_j^0 an open set $V_j \subset \mathrm{dmn}(g_j)$ such that $y_j^0 \in V_j$ and $V_j \setminus \{g_j \neq 0\}$ is of the first category. If for some $1 \leq j \leq n$ we have $y_j \in V_j \setminus \{g_j \neq 0\}$, let us consider (FE) not satisfied. Still we have (FE) is satisfied except a set of first category. No let us apply again Proposition 2 of our main tool. We obtain that x_i^0 is in essence almost inner point of $\mathrm{dmn}(f_i)$.

Using the symmetry of the functional equation, we obtain that if non of the sets $\{g_j \neq 0\}$, $1 \leq j \leq n$ is of first category and $\{g_j \neq 0\}$ has second category at a point, then this point is in essence an inner point of the corresponding set and the sets $\{f_i \neq 0\}$, $1 \leq i \leq m$ has of the second category for $1 \leq j \leq n$.

Now the statement of the theorem follows: The set of points at which $\{f_i \neq 0\}$ is of first category is an open set and any point at which $\{f_i \neq 0\}$ is of second category is in essence an innner point of this set, hence $\{f_i = 0\}$ is of first category in it, so this is a nonempty open set. Because $\mathrm{dmn}(f_i)$ is open and connected, we obtain that $\{f_i = 0\}$ is of first category at each point of $\mathrm{dmn}(f_i)$. This means that $\{f_i = 0\}$ is of the first category for $1 \leq i \leq m$. Similarly, $\{g_j = 0\}$ is of the first category for $1 \leq j \leq n$.

As a next step, we want to deduce that solutions having Baire property and satifying (FE) except a set of first category are equal to continuous solutions of (FE) except a set of the first category. The following theorem easily follows from Theorem 10.4 in [5] (see also (1) of the Remark 10.5 in [5]):

Theorem 3. *Let Z be a regular space, Z_i $(i = 1, 2, \dots, n)$ topological spaces, and T a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$ $(i = 1, 2, \dots, n)$, D an open subset of $T \times Y$, and $W \subset D \times Z_1 \times \dots \times Z_n$. Let $T' \subset T$ be a dense subset, $f : T' \to Z$, $g_i : D \to X_i$, and $h : W \to Z$. Suppose that the function f_i is defined except of a set of the first category on X_i with values in Z_i $(i = 1, 2, \dots, n)$ and the following conditions are satisfied:*

(1) for all $t \in T'$ for all $y \in D_t$ except of a set of first category we have

$$\Big(t, y, f_1\big(g_1(t, y)\big), \dots, f_n\big(g_n(t, y)\big)\Big) \in W$$

and

$$f(t) = h\Big(t, y, f_1\big(g_1(t, y)\big), \dots, f_n\big(g_n(t, y)\big)\Big);$$

(2) for each fixed y in Y, the function h is continuous in the other variables;

(3) f_i has the Lusin–Baire property on its domain $(i = 1, 2, \dots, n)$;

(4) g_i and the partial derivative $\dfrac{\partial g_i}{\partial y}$ is continuous on D $(i = 1, 2, \dots, n)$;

(5) *for each* $t \in T$ *there exists a* y *such that* $(t, y) \in D$ *and the partial derivative* $\dfrac{\partial g_i}{\partial y}$ *has rank* r_i *at* $(t, y) \in D$ $(i = 1, 2, \dots, n)$.

Then f *has a unique continuous extension to* T.

Theorem 4. *Let us suppose that the complex valued functions f_i, $0 \le i \le k$ and g_j, $0 \le j \le n$ have Baire property and (FE) is satisfied except a set of first category, where the inner functions satisfies (1)–(4) of the key observations. Then either on both sides one of the functions is zero except a set of first category, or except a set of first category the functions f_i, $0 \le i \le k$ and g_j, $0 \le j \le n$ are equal to complex valued nonzero continuous functions satisfying (FE) everywhere.*

Proof. Applying the zero–one–law to the absolute value of the signum of the functions in (FE) we get that if on one side non of the functions is everywhere zero except a set of first category, then all of them are everywhere nonzero except a set of first category. We may rewrite the functional equation in the form

$$f_0(y) = \frac{g_0(x)\prod_{i=1}^{k} f_i(w_i)}{\prod_{j=1}^{n} g_j(z_j),}$$

which is satisfied for all $y \in \Delta_{n-1}$ except a set of first category for all $z \in \Delta_{k-1}^{n}$ except a set of first category. Let all Z's in the previous theorem be $\mathbb{C} \setminus \{0\}$. Applying the previous theorem we obtain that f_0 can be uniquely extended to a continuous function $\tilde{f}_0 : \Delta_{n-1} \to \mathbb{C} \setminus \{0\}$; of course, replacing f_0 by $\tilde{f}_0$, the functional equation still satisfied except a set of first category. Applying this step again and again, all the functions f_i, $0 \leq i \leq k$ and g_j, $0 \leq j \leq n$ may be replaced by continuous nonzero functions $\tilde{f}_i$, $0 \leq i \leq k$ and $\tilde{g}_j$, $0 \leq j \leq n$ so that the functional equation (FE) will still be satisfied except a set of first category. But since then both sides of the functional equation are continuous functions on an open set and they are equal on a dense subset, they are equal everywhere. Hence (FE) is satisfied by the continuous functions $\tilde{f}_i$, $0 \leq i \leq k$ and $\tilde{g}_j$, $0 \leq j \leq n$ everywhere.

Theorem 5. *Let us suppose that the real valued functions f_i, $0 \leq i \leq k$ and g_j, $0 \leq j \leq n$ have Baire property and (FE) is satisfied except a set of first category, where the inner functions satisfies (1)–(4) in the key observations and moreover are in $\mathcal{C}^\infty$. Then either on both sides one of the functions is zero except a set of first category, or the functions f_i, $0 \leq i \leq k$ and g_j, $0 \leq j \leq n$ are equal except a set of first category to real valued nonzero $\mathcal{C}^\infty$-functions satisfying (FE) everywhere.*

Proof. Just like in the proof of the previous theorem, we obtain that if on one side non of the functions is zero except a set of first category, then they may be replaced by continuous real valued nonzero functions $\tilde{f}_i$, $0 \leq i \leq k$ and $\tilde{g}_j$, $0 \leq j \leq n$ such that (FE) is satisfied everywhere. As in [6] (or in [5], Theorem 23.4) we obtain from general results that $|\tilde{f}_i|$, $0 \leq i \leq k$ and $|\tilde{g}_j|$, $0 \leq j \leq n$ are in $\mathcal{C}^\infty$. But since the domains of the continuous functions $\tilde{f}_i$ and $\tilde{g}_j$ are connected, they do not change sign. Hence $\tilde{f}_i = \pm|\tilde{f}_i|$ and $\tilde{g}_j = \pm|\tilde{g}_j|$, i.e. they are in $\mathcal{C}^\infty$.

Theorem 6. *Solutions having the property of Baire and not constant on the complement of some first category set of the functional equation*

$$\begin{gathered}\Big(f\big(t(x+y)\big)-f(tx)\Big)\Big(f(x+y)-f(y)\Big)= \\ \Big(f\big(t(x+y)\big)-f(ty)\Big)\Big(f(x+y)-f(x)\Big)\end{gathered} \tag{1}$$

$(t,x,y>0)$ are continuous.

Proof. We prove that by solution which is nonconstant on the complement of any set having first category the term

$$f\big(t(x+y)\big)-f(ty) \tag{2}$$

is nonzero for a "large" set of triples (x,t,y), hence we may divide by it and apply the general methods of [5], expecially a variant of the theorem 9.1. We shall use also the ideas of the proofs of 4.1, 4.2, 4.3 from [5], but for simplicity we shall give here a self-stending proof, not using the results of [5], only the ideas.

If a function $f :]0,+\infty[\to \mathbb{R}$ has the property of Baire then there exists a set $E \subset]0,+\infty[$ having complement of the first category such that $f|_E$ is continuous. The function $f|_E$ cannot be constant, hence there exists c^- and c^+ in E such that $f(c^-) \neq f(c^+)$. Without restricting generality we may suppose that $c^- < c^+$ and $f(c^-) < f(c^+)$ because otherwise we may consider the function $-f$ instead of f satisfying the same functional equation (1). Let $c := c^+ - c^-$. Let $x_0 > 0$ arbitrary. We shall prove that f is continuous at x_0.

Let us choose disjoint open neighbourhoods $V^- \subset]0, +\infty[$ and $V^+ \subset]0, +\infty[$ of c^+ and c^-, respectively, such that each element of $f(E \cap V^-)$ is less then any element of $f(E \cap V^+)$. Let X, Y and T be a neighbourhoods of x_0, c/x_0 and $x_0 c^-/c$, respectively, such that for each $x \in X$, $y \in Y$ and $t \in T$ we have $ty \in V^-$ and $t(x+y) \in V^+$. For x_0 let us choose a $t_0 \in T$ such that $t_0 x_0 \in E$ and then a $y_0 \in Y$ such that $y_0 \in E$, $x_0 + y + 0 \in E$, $t_0 y_0 \in E$ and $t_0(x_0 + y_0) \in E$. Because for $z := f(x_0)$, $z_1 := f(x_0 + y_0)$, $z_2 := f(y_0)$, $z_3 := f(t_0 x_0)$, $z_4 := f(t_0 y_0)$ and $z_5 := f\big(t_0(x_0 + y_0)\big)$ we have $z_4 \neq z_5$ and by the functional equation

$$z = z_1 + (z_2 - z_1)\frac{z_4 - z_3}{z_5 - z_4},$$

for an arbitrary neighbourhood W of z there exit neighbourhoods W_1, W_2, W_3, W_4 and W_5 of z_1,

z_2, z_3, z_4 and z_5, respectively, such that for any $z_1' \in W_1$, $z_2' \in W_2$, $z_3' \in W_3$, $z_4' \in W_4$ and $z_5' \in W_5$ we have $w_4' \neq w_5'$ and

$$z' = z_1' + (z_2' - z_1')\frac{z_4' - z_3'}{z_5' - z_4'} \in W.$$

Because $f|_E$ is continuous at z_i, there exist V_i such that for $z_i' \in E \cap V_i$ we have $f(z_i') \in W_i$ for $i = 1, 2, 3, 4, 5$. Now there exist neighbourhoods X_0, Y_0 and T_0 of x_0, y_0 and t_0, respectively, contained in X, Y and T, such that $x + y \in V_1$, $y \in V_2$, $tx \in V_3$, $ty \in V_4$ and $t(x + y) \in V_5$ whenever $x \in X_0$, $y \in Y_0$ and $t \in T_0$. Fixing any $x \in X_0$ let us choose a $t \in T_0$ such that $tx \in E$. Fixing this, too, let us choose a $y \in Y_0$ such that $y \in E$, $x + y \in E$, $ty \in E$ and $t(x + y) \in E$. Because for $z' := f(x)$, $z_1' := f(x+y)$, $z_2' := f(y)$, $z_3' := f(tx)$, $z_4' := f(ty)$ and $z_5' := f\bigl(t(x+y)\bigr)$ we have $z_4' \neq z_5'$ and by the functional equation

$$z' = z_1' + (z_2' - z_1')\frac{z_4' - z_3'}{z_5' - z_4'} \in W.$$

This proves that f is continuous at x_0.

Remark. Solutions of equation (1) constant everywhere except a set of first category are constant everywhere. This follows easily.

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