Bevezetés a matematikába

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Ezek a programok csak szemléltetésre szolgálnak.

1. Halmazok

2. Természetes számok

3. A számfogalom bővítése

```
> restart;
```

3.1. Egész számok

3.1.1. Osztályozás kompatibilitása művelettel.

```
> `type/ordpair`:=proc(x) type(x,list) and nops(x)=2 end;
```

$$type/ordpair := \mathbf{proc}(x)\ type(x, list)\ \mathbf{and}\ nops(x) = 2\ \mathbf{end\ proc} \qquad (3.1.1.1)$$

```
> iscompbinop:=proc(X::set,E::set(ordpair),f::procedure)
  local x,xx,y,yy;
  for x in X do for y in X do
    if not f(x,y) in X then return false fi;
    for xx in X do for yy in X do
      if [x,xx] in E and [y,yy] in E and not [f(x,y),f(xx,yy)
  ] in E
        then return false fi;
  od; od; od; od; true; end;

  X:={0,1,2,3,4,5}; E:={[0,0],[0,3],[3,0],[3,3],[1,1],[1,4],
  [4,1],[4,4],[2,2],[2,5],[5,2],[5,5]}; f:=(x,y)->irem(x+y,6)
  ; iscompbinop(X,E,f);
```

$$iscompbinop := \mathbf{proc}(X::set, E::(set(ordpair)), f::procedure)$$
$$\quad \mathbf{local}\ x, xx, y, yy;$$
$$\quad \mathbf{for}\ x\ \mathbf{in}\ X\ \mathbf{do}$$
$$\quad\quad \mathbf{for}\ y\ \mathbf{in}\ X\ \mathbf{do}$$
$$\quad\quad\quad \mathbf{if\ not}\ in(f(x, y), X)\ \mathbf{then}$$
$$\quad\quad\quad\quad \mathbf{return}\ false$$
$$\quad\quad\quad \mathbf{end\ if};$$

```
            for xx in X do
                for yy in X do
                    if in([x, xx], E) and in([y, yy],
                    E) and not in([f(x, y), f(xx, yy)], E) then
                        return false
                    end if
                end do
            end do
        end do
    end do;
    true
end proc
```

$$X := \{0, 1, 2, 3, 4, 5\}$$

$$E := \{[0, 0], [0, 3], [3, 0], [3, 3], [1, 1], [1, 4], [4, 1], [4, 4], [2, 2], [2, 5], [5, 2], [5, 5]\}$$

$$f := (x, y) \rightarrow irem(x + y, 6)$$

$$true \tag{3.1.1.2}$$

3.1.2. Példa.

3.1.3. Feladat.

3.1.4. Osztályozás kompatibilitása relációval.

```
> iscomprel:=proc(X::set,E::set(ordpair),R::set(ordpair))
  local x,xx,y,yy;
  for x in X do for y in X do
    for xx in X do for yy in X do
      if [x,xx] in E and [y,yy] in E and [x,y] in R and not
  [xx,yy] in R
        then return false fi;
  od; od; od; od; true; end;

  X:={0,1,2,3}; E:={[0,0],[0,2],[2,0],[2,2],[1,1],[1,3],[3,1]
  ,[3,3]};
  R:={[0,1],[0,3],[2,1],[2,3]}; iscomprel(X,E,R);
```

$iscomprel := \mathbf{proc}(X::set, E::(set(ordpair)), R::(set(ordpair)))$

```
    local x, xx, y, yy;
    for x in X do
        for y in X do
            for xx in X do
                for yy in X do
```

```
                    if in([x, xx], E) and in([y, yy],
                    E) and in([x, y], R) and  not in([xx, yy], R) then
                        return false
                    end if
                end do
            end do
        end do
    end do;
    true
end proc
```

$$X := \{0, 1, 2, 3\}$$
$$E := \{[0, 0], [3, 3], [1, 1], [2, 2], [0, 2], [1, 3], [2, 0], [3, 1]\}$$
$$R := \{[0, 3], [0, 1], [2, 1], [2, 3]\}$$
$$true \tag{3.1.4.1}$$

3.1.5. Példa.

3.1.6. Tétel.

```
> type(5,integer); type(-3,integer); type(0,integer);
```

$$true$$
$$true$$
$$true \tag{3.1.6.1}$$

```
> `&~`:=(x,y)->x[1]+y[2]=x[2]+y[1];

  `&+`:=(x,y)->[x[1]+y[1],x[2]+y[2]];

  `&*`:=(x,y)->[x[1]*y[1]+x[2]*y[2],x[1]*y[2]+y[1]*x[2]];

  `&le`:=(x,y)->x[1]+y[2]<=x[2]+y[1];

  [7,4]&~[3,0]; [7,4]&+[2,6]; [2,1]&*[2,4]; [3,5]&le[2,3];
```

$$\&\sim := (x, y) \to x_1 + y_2 = x_2 + y_1$$
$$\&+ := (x, y) \to [x_1 + y_1, x_2 + y_2]$$
$$\&* := (x, y) \to [x_1\, y_1 + x_2\, y_2, x_1\, y_2 + y_1\, x_2]$$
$$\&le := (x, y) \to x_1 + y_2 \le x_2 + y_1$$
$$7 = 7$$
$$[9, 10]$$
$$[8, 10] \tag{3.1.6.2}$$

$$6 \leq 7 \tag{3.1.6.2}$$

3.1.7. Tétel: ℕ beágyazása ℤ-be.

3.1.8. Az egész számok rendezése.

3.1.9. Az egész számok szorzása.

3.1.10. Az egész számok számítógépes ábrázolása.

3.1.11. Hatványozás egész kitevővel.

```
> (a^(-1))^5; a^(m+n); expand(%); (a*b)^5;
  (a^n)^m; combine(%,power) assuming integer;
```

$$\frac{1}{a^5}$$

$$a^{m+n}$$

$$a^m\, a^n$$

$$a^5\, b^5$$

$$\left(a^n\right)^m$$

$$a^{n\,m} \tag{3.1.11.1}$$

3.1.12. Példa.

3.1.13. Gyűrűk.

```
> isgrupoid:=proc(G::set,f::procedure) local x,y;
  for x in G do for y in G do if not f(x,y) in G then return
  false fi;
  od; od; true; end;
```

```
isgrupoid := proc(G::set, f::procedure)                    (3.1.13.1)
    local x, y;
    for x in G do
        for y in G do
            if not in(f(x, y), G) then
                return false
            end if
        end do
    end do;
    true
end proc
```

```
> neutral:=proc(G::set,f::procedure) local x,y,s,S;
  if not isgrupoid(G,f) then return NULL fi;
```

```
   for x in G do s:=true; for y in G do
     if f(x,y)<>y or f(y,x)<>y then s:=false; break; fi;
   od; if s then return x fi; od; NULL end;

   G:={a,b,c};neutral(G,(x,y)->y);neutral(G,(x,y)->y);

   neutral({0,1,2},(x,y)->irem(x+y,3));
```

```
neutral := proc(G::set, f::procedure)
    local x, y, s, S;
    if not isgrupoid(G, f) then
        return NULL
    end if;
    for x in G do
        s := true;
        for y in G do
            if f(x, y) <> y or f(y, x) <> y then
                s := false;
                break
            end if
        end do;
        if s then
            return x
        end if
    end do;
    NULL
end proc
```

$$G := \{a, b, c\}$$

$$0 \tag{3.1.13.2}$$

```
> issemigroup:=proc(G::set,f::procedure) local x,y,z;
   if not isgrupoid(G,f) then return false fi;
   for x in G do for y in G do for z in G do
     if f(x,f(y,z))<>f(f(x,y),z) then return false fi;
   od; od; od; true end;

   issemigroup({a,b,c},(x,y)->x);

   issemigroup({true,false},(x,y)-> x implies y);
```

```
issemigroup := proc(G::set, f::procedure)
    local x, y, z;
    if not isgrupoid(G, f) then
```

```
        return false
    end if;
    for x in G do
        for y in G do
            for z in G do
                if f(x, f(y, z)) <> f(f(x, y), z) then
                    return false
                end if
            end do
        end do
    end do;
    true
end proc
```

$$\begin{array}{l} true \\ false \end{array} \tag{3.1.13.3}$$

```
> isgroup:=proc(G::set,f::procedure) local x,y,n,i;
  if not isgrupoid(G,f) then return false fi;
  if not issemigroup(G,f) then return false fi;
  n:=neutral(G,f); if n=NULL then return false fi;
  for x in G do i:=false; for y in G do
    if f(x,y)=n and f(y,x)=n then i:=true; break fi;
  od; if i=false then return false fi; od; true; end;

  isgroup({0,1,2},(x,y)->irem(x+y,3));
```

```
isgroup := proc(G::set, f::procedure)
    local x, y, n, i;
    if not isgrupoid(G, f) then
        return false
    end if;
    if not issemigroup(G, f) then
        return false
    end if;
    n := neutral(G, f);
    if n = NULL then
        return false
    end if;
    for x in G do
        i := false;
        for y in G do
```

```
            if f(x, y) = n and f(y, x) = n then
                i := true;
                break
            end if
        end do;
        if i = false then
            return false
        end if
    end do;
    true
end proc
```

$$true \tag{3.1.13.4}$$

```
> iscommutative:=proc(G::set,f::procedure) local x,y;
  if not isgrupoid(G,f) then return false fi;
  for x in G do for y in G do
    if f(x,y)<>f(y,x) then return false fi;
  od; od; true; end;

  iscommutative({0,1,2},(x,y)->irem(x+y,3));
```

```
iscommutative := proc(G::set, f::procedure)
    local x, y;
    if not isgrupoid(G, f) then
        return false
    end if;
    for x in G do
        for y in G do
            if f(x, y) <> f(y, x) then
                return false
            end if
        end do
    end do;
    true
end proc
```

$$true \tag{3.1.13.5}$$

```
> isabeliangroup:=proc(G::set,f::procedure)
  isgroup(G,f) and iscommutative(G,f) end;

  iscommutative({0,1,2},(x,y)->irem(x+y,3));
```

```
isabeliangroup := proc(G::set, f::procedure)
    isgroup(G, f) and iscommutative(G, f)
end proc
```

true (3.1.13.6)

```
> isleftdistributive:=proc(R::set,f::procedure,g::procedure)
  local x,y,z;
  if not isgrupoid(R,f) then return false fi;
  if not isgrupoid(R,g) then return false fi;
  for x in R do for y in R do for z in R do
    if g(x,f(y,z))<>f(g(x,y),g(x,z)) then return false fi;
  od; od; od; true end;
```

```
isleftdistributive := proc(R::set, f::procedure, g::procedure)
    local x, y, z;
    if not isgrupoid(R, f) then
        return false
    end if;
    if not isgrupoid(R, g) then
        return false
    end if;
    for x in R do
        for y in R do
            for z in R do
                if g(x, f(y, z)) <> f(g(x, y), g(x, z)) then
                    return false
                end if
            end do
        end do
    end do;
    true
end proc
```

(3.1.13.7)

```
> isrightdistributive:=proc(R::set,f::procedure,g::procedure)
  local x,y,z;
  if not isgrupoid(R,f) then return false fi;
  if not isgrupoid(R,g) then return false fi;
  for x in R do for y in R do for z in R do
    if g(f(y,z),x)<>f(g(y,x),g(z,x)) then return false fi;
  od; od; od; true end;
```

```
isrightdistributive := proc(R::set, f::procedure, g::procedure)
    local x, y, z;
    if not isgrupoid(R, f) then
        return false
```

(3.1.13.8)

```
    end if;
    if not isgrupoid(R, g) then
        return false
    end if;
    for x in R do
        for y in R do
            for z in R do
                if g(f(y, z), x) <> f(g(y, x), g(z, x)) then
                    return false
                end if
            end do
        end do
    end do;
    true
end proc
```

```
> isring:=proc(R::set,f::procedure,g::procedure)
  isabeliangroup(R,f) and issemigroup(R,g)
  and isleftdistributive(R,f,g) and isrightdistributive(R,f,
  g) end;
```

$$isring := \mathbf{proc}(R::set, f::procedure, g::procedure) \quad (3.1.13.9)$$

$$isabeliangroup(R, f) \ \mathbf{and}\ issemigroup(R, g) \ \mathbf{and}\ isleftdistributive(R, f, g) \ \mathbf{and}\ isrightdistributive(R, f, g)$$

```
end proc
```

```
> iscommutativering:=proc(R::set,f::procedure,g::procedure)
  isring(R,f,g) and iscommutative(R,g) end;
```

$$iscommutativering := \mathbf{proc}(R::set, f::procedure, g::procedure) \quad (3.1.13.10)$$

$$isring(R, f, g) \ \mathbf{and}\ iscommutative(R, g)$$

```
end proc
```

```
> isringwithunity:=proc(R::set,f::procedure,g::procedure)
  isring(R,f,g) and neutral(R,g)<>NULL end;
```

$$isringwithunity := \mathbf{proc}(R::set, f::procedure, g::procedure) \quad (3.1.13.11)$$

$$isring(R, f, g) \ \mathbf{and}\ neutral(R, g) <> NULL$$

```
end proc
```

```
> isringwithunity({0},(x,y)->0,(x,y)->0);
```

$$true \quad (3.1.13.12)$$

```
> X:={a,b,c}; P:=combinat[powerset](X);
  isring(P,(x,y)->symmdiff(x,y),(x,y)->{});
```

$$X := \{a, b, c\}$$

$$P := \{\{\}, \{a, b, c\}, \{b, c\}, \{c\}, \{a, c\}, \{a\}, \{b\}, \{a, b\}\}$$

$$true \quad (3.1.13.13)$$

3.1.14. Példa.

```
> iscommutativering(P,(x,y)->symmdiff(x,y),(x,y)->x intersect
  y);

  isringwithunity(P,(x,y)->symmdiff(x,y),(x,y)->x intersect
  y);
```

true

true (3.1.14.1)

->3.1.15. Feladat.

3.1.16. Példák.

->3.1.17. Feladat.

->3.1.18. Feladat.

->3.1.19. Feladat.

3.1.20. Az általános disztributivitás tétele.

```
> A:=sum(a[i],i=1..4); B:=sum(b[j],j=1..5); A*B; expand(%);
```

$$A := a_1 + a_2 + a_3 + a_4$$

$$B := b_1 + b_2 + b_3 + b_4 + b_5$$

$$\left(a_1 + a_2 + a_3 + a_4\right)\left(b_1 + b_2 + b_3 + b_4 + b_5\right)$$

$$a_1 b_1 + a_1 b_2 + a_1 b_3 + a_1 b_4 + a_1 b_5 + a_2 b_1 + a_2 b_2 + a_2 b_3 + a_2 b_4 + a_2 b_5 + a_3 b_1 + a_3 b_2 + a_3 b_3 + a_3 b_4 + a_3 b_5 + a_4 b_1 + a_4 b_2 + a_4 b_3 + a_4 b_4 + a_4 b_5 \quad (3.1.20.1)$$

3.1.21. Nullosztók, integritási tartomány, rendezett integritási tartomány.

3.1.22. Tétel.

3.1.23. Tétel.

->3.1.24. Feladat.

->3.1.25. Feladat.

3.2. Racionális számok

3.2.1. Tétel.

```
> type(5/7,rational); type(0,rational);
```

$$true$$

$$true \tag{3.2.1.1}$$

```
> `&~`:=(x,y)->x[1]*y[2]=y[1]*x[2];

  `&+`:=(x,y)->[x[1]*y[2]+x[2]*y[1],x[2]*y[2]];

  `&*`:=(x,y)->[x[1]*y[1],x[2]*y[2]];

  `&le`:=(x,y)->(y[1]*x[2]-y[2]*x[1])*x[2]*y[2]>=0;

  [1,5]&~[2,10]; [1,2]&+[2,3]; [1,2]&*[2,3]; [1,2]&le[2,3];
```

$$\&\sim := (x, y) \to x_1\, y_2 = y_1\, x_2$$

$$\&+ := (x, y) \to \left[x_1\, y_2 + y_1\, x_2,\ x_2\, y_2\right]$$

$$\&* := (x, y) \to \left[x_1\, y_1,\ x_2\, y_2\right]$$

$$\&le := (x, y) \to 0 \le \left(y_1\, x_2 - x_1\, y_2\right) x_2\, y_2$$

$$10 = 10$$

$$[7, 6]$$

$$[2, 6]$$

$$0 \le 6 \tag{3.2.1.2}$$

3.2.2. Tétel: $\mathbb{Z}$ beágyazása $\mathbb{Q}$-ba.

3.2.3. Ferdetest, test, rendezett test.

```
> isskewfield:=proc(R::set,f::procedure,g::procedure) local
  n;
  n:=neutral(R,f); if n=NULL then return false fi;
  isring(R,f,g) and isgroup(R minus {n},g) end;
```

isskewfield := **proc**(*R*::*set*, *f*::*procedure*, *g*::*procedure*) (3.2.3.1)

local *n*;
n := *neutral*(*R*, *f*);
if *n* = *NULL* **then**
return *false*
end if;
isring(*R*, *f*, *g*) **and** *isgroup*(*minus*(*R*, {*n*}), *g*)
end proc

```
> isfield:=proc(R::set,f::procedure,g::procedure) local n;
  n:=neutral(R,f); if n=NULL then return false fi;
```

```
   isring(R,f,g) and isabeliangroup(R minus {n},g) end;
```

```
isfield := proc(R::set, f::procedure, g::procedure)                    (3.2.3.2)
    local n;
    n := neutral(R, f);
    if n = NULL then
        return false
    end if;
    isring(R, f, g) and isabeliangroup(minus(R, {n}), g)
end proc
```

```
> &+(0,0):=0; &+(0,1):=1; &+(1,0):=1; &+(1,1):=0;

  &*(0,0):=0; &*(0,1):=0; &*(1,0):=0; &*(1,1):=1;
```

$$0 \,\&+\, 0 := 0$$
$$0 \,\&+\, 1 := 1$$
$$1 \,\&+\, 0 := 1$$
$$1 \,\&+\, 1 := 0$$
$$0 \,\&^{*}\, 0 := 0$$
$$0 \,\&^{*}\, 1 := 0$$
$$1 \,\&^{*}\, 0 := 0$$
$$1 \,\&^{*}\, 1 := 1 \qquad (3.2.3.3)$$

```
> isfield({0,1},(x,y)->x&+y,(x,y)->x&*y);
```

$$true \qquad (3.2.3.4)$$

3.2.4. Példák.

```
> `&+`:=(x,y)->irem(x+y,5); `&*`:=(x,y)->irem(x*y,5); 3&+4;
  3&*4;
```

$$\&+ := (x, y) \to irem(x + y, 5)$$
$$\&^{*} := (x, y) \to irem(x\,y, 5)$$
$$2$$
$$2 \qquad (3.2.4.1)$$

3.2.5 Tétel: Q beágyazása rendezett testbe.

->3.2.6. Feladat.

->3.2.7. Feladat.

->3.2.8. Feladat.

3.2.9. Feladat.

3.3. Valós számok

```
> restart;
```

3.3.1. Állítás.

```
> i:='i': x:=0:
  for i from 0 do while (x+1)^2<2*10^(2*i) do x:=x+1: od; x;
  x:=x*10: od;
```

$$1$$

$$x := 10$$

$$14$$

$$x := 140$$

$$141$$

$$x := 1410$$

$$1414$$

$$x := 14140$$

$$14142$$

$$x := 141420$$

$$141421$$

$$x := 1414210$$

$$1414213$$

$$x := 14142130$$

$$14142135$$

$$x := 141421350$$

$$141421356$$

$$x := 1414213560$$

$$1414213562$$

$$x := 14142135620$$

$$14142135623$$

$$x := 141421356230$$

$$141421356237$$

$$x := 1414213562370$$

$$1414213562373$$

$$x := 14142135623730$$

$$14142135623730$$

$$x := 141421356237300$$

$$141421356237309$$

$$x := 1414213562373090$$

1414213562373095

$x := 14142135623730950$

14142135623730950

$x := 141421356237309500$

141421356237309504

$x := 1414213562373095040$

1414213562373095048

$x := 14142135623730950480$

14142135623730950488

$x := 141421356237309504880$

14142135623730950488

$x := 1414213562373095048800$

14142135623730950488 01

$x := 141421356237309504880 10$

14142135623730950488 016

$x := 1414213562373095048801 60$

14142135623730950488 0168

$x := 141421356237309504880 1680$

14142135623730950488 01688

$x := 1414213562373095048801 6880$

14142135623730950488 016887

$x := 1414213562373095048801 68870$

14142135623730950488 016887 2

$x := 1414213562373095048801 688720$

14142135623730950488 016887 24

$x := 1414213562373095048801 6887240$

14142135623730950488 016887 242

$x := 1414213562373095048801 68872420$

14142135623730950488 016887 2420

$x := 1414213562373095048801 688724200$

14142135623730950488 016887 2420 9

$x := 1414213562373095048801 6887242090$

14142135623730950488 016887 2420 96

$x := 1414213562373095048801 68872420960$

14142135623730950488 016887 2420 969

$x := 1414213562373095048801 688724209690$

14142135623730950488 016887 2420 9698

$x := 1414213562373095048801 6887242096980$

141421356237309504880168872420969800

$x :=$ 1414213562373095048801688724209698000

141421356237309504880168872420969807

$x :=$ 1414213562373095048801688724209698070

141421356237309504880168872420969807 8

$x :=$ 14142135623730950488016887242096980780

141421356237309504880168872420969807 85

$x :=$ 14142135623730950488016887242096980785 0

141421356237309504880168872420969807 856

$x :=$ 14142135623730950488016887242096980785 60

141421356237309504880168872420969807 8569

$x :=$ 14142135623730950488016887242096980785690

141421356237309504880168872420969807 85696

$x :=$ 14142135623730950488016887242096980785696 0

141421356237309504880168872420969807 856967

$x :=$ 14142135623730950488016887242096980785696 70

141421356237309504880168872420969807 856967 1

$x :=$ 14142135623730950488016887242096980785696 710

141421356237309504880168872420969807 856967 18

$x :=$ 14142135623730950488016887242096980785696 7180

141421356237309504880168872420969807 856967 187

$x :=$ 14142135623730950488016887242096980785696 71870

141421356237309504880168872420969807 856967 1875

$x :=$ 14142135623730950488016887242096980785696 718750

141421356237309504880168872420969807 856967 18753

$x :=$ 14142135623730950488016887242096980785696 7187530

141421356237309504880168872420969807 856967 187537

$x :=$ 14142135623730950488016887242096980785696 71875370

141421356237309504880168872420969807 856967 1875376

$x :=$ 14142135623730950488016887242096980785696 718753760

141421356237309504880168872420969807 856967 1875376 9

$x :=$ 14142135623730950488016887242096980785696 7187537690

141421356237309504880168872420969807 856967 1875376 94

$x :=$ 14142135623730950488016887242096980785696 71875376940

141421356237309504880168872420969807 856967 1875376 948

$x :=$ 14142135623730950488016887242096980785696 718753769480

141421356237309504880168872420969807 856967 1875376 9480

$x :=$ 14142135623730950488016887242096980785696 7187537694800

$$
\begin{aligned}
&14142135623730950488016887242096980785696718753769480731\\
&x := 141421356237309504880168872420969807856967187537694807\\
&14142135623730950488016887242096980785696718753769480731\\
&x := 1414213562373095048801688724209698078569671875376948073\\
&14142135623730950488016887242096980785696718753769480731\\
&x := 14142135623730950488016887242096980785696718753769480730\\
&14142135623730950488016887242096980785696718753769480731\\
&x := 141421356237309504880168872420969807856967187537694807310\\
&14142135623730950488016887242096980785696718753769480731\\
&x := 1414213562373095048801688724209698078569671875376948073170\\
&14142135623730950488016887242096980785696718753769480731\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 1760\\
&14142135623730950488016887242096980785696718753769480731766\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 17660\\
&14142135623730950488016887242096980785696718753769480731\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 176670\\
&14142135623730950488016887242096980785696718753769480731\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 1766790\\
&1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 7\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 17667970\\
&1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 73\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 176679730\\
&1414213562373095048801688724209698078569671875376948073\backslash\\
&\qquad 737\\
&x := \\
&\qquad 1414213562373095048801688724209698078569671875376948073\backslash
\end{aligned}
$$

1766797370
14142135623730950488016887242096980785696718753769480731766\
7379
$x :=$
14142135623730950488016887242096980785696718753769480731\
17667973790
14142135623730950488016887242096980785696718753769480731766\
73799
$x :=$
14142135623730950488016887242096980785696718753769480731\
176679737990
14142135623730950488016887242096980785696718753769480731766\
737990
$x :=$
14142135623730950488016887242096980785696718753769480731\
1766797379900
14142135623730950488016887242096980785696718753769480731766\
7379907
$x :=$
14142135623730950488016887242096980785696718753769480731\
17667973799070
14142135623730950488016887242096980785696718753769480731766\
73799073
$x :=$
14142135623730950488016887242096980785696718753769480731\
176679737990730
14142135623730950488016887242096980785696718753769480731766\
737990732
$x :=$
14142135623730950488016887242096980785696718753769480731\
1766797379907320
14142135623730950488016887242096980785696718753769480731766\
7379907324
$x :=$
14142135623730950488016887242096980785696718753769480731\
17667973799073240
14142135623730950488016887242096980785696718753769480731766\
73799073247

```
x:=
    14142135623730950488016887242096980785696718753769480731766797379907324701414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    141421356237309504880168872420969807856967187537694807317667973799073247801414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    1414213562373095048801688724209698078569671875376948073176679737990732478401414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    14142135623730950488016887242096980785696718753769480731766797379907324784601414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    141421356237309504880168872420969807856967187537694807317667973799073247846201414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    1414213562373095048801688724209698078569671875376948073176679737990732478462101414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    14142135623730950488016887242096980785696718753769480731766797379907324784621001414213562373095048801688724209698078569671875376948073176679737990732478
x:=
    141421356237309504880168872420969807856967187537694807317667973799073247846210701414213562373095048801688724209698078569671875376948073176679737990732478
```

1414213562373095048801688724209698078569671875376948073176679\
7379907324784621070

$x:=$
1414213562373095048801688724209698078569671875376948073\
17667973799073247846210700

1414213562373095048801688724209698078569671875376948073176679\
73799073247846210703

$x:=$
1414213562373095048801688724209698078569671875376948073\
176679737990732478462107030

1414213562373095048801688724209698078569671875376948073176679\
737990732478462107038

$x:=$
1414213562373095048801688724209698078569671875376948073\
1766797379907324784621070380

1414213562373095048801688724209698078569671875376948073176679\
7379907324784621070388

$x:=$
1414213562373095048801688724209698078569671875376948073\
17667973799073247846210703880

1414213562373095048801688724209698078569671875376948073176679\
73799073247846210703885

$x:=$
1414213562373095048801688724209698078569671875376948073\
176679737990732478462107038850

1414213562373095048801688724209698078569671875376948073176679\
737990732478462107038850

$x:=$
1414213562373095048801688724209698078569671875376948073\
1766797379907324784621070388500

1414213562373095048801688724209698078569671875376948073176679\
7379907324784621070388503

$x:=$
1414213562373095048801688724209698078569671875376948073\
17667973799073247846210703885030

1414213562373095048801688724209698078569671875376948073176679\
73799073247846210703885038

$x:=$

```
      141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      80
141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      7379907324784621070388503870
x:=
      141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      80
141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      7379907324784621070388503875
x:=
      141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      80
141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      7379907324784621070388503753
x:=
      141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      80
141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      7379907324784621070388503534
x:=
      141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      80
141421356237309504880168872420969807856967187537694807317667973799073247846210703885038\
      7379907324784621070388503343
Warning,  computation interrupted
```

3.3.2. Arkhimédészi tulajdonság.

3.3.3. Állítás.

3.3.4. Állítás.

**3.3.5. Tétel.*

3.3.6. Tétel.

3.3.7. Valós számok.

```
> abs(7.4); abs(-3); abs(0); signum(7.4); signum(-3); signum
  (0);
                                  7.4
                                   3
                                   0
```

$$1$$
$$-1$$
$$0 \qquad (3.3.7.1)$$

```
> floor(3.14); ceil(3.14); ceil(-3.14);
```

$$3$$
$$4$$
$$-3 \qquad (3.3.7.2)$$

```
> Rmod:=proc(x::realcons,y::realcons) if y=0 then x else x-
  floor(x/y)*y fi; end;

  Rmod(5,0); Rmod(3.1415,2.78);
```

$$Rmod := \mathbf{proc}(x::realcons,\ y::realcons)$$
$$\quad \mathbf{if}\ y = 0\ \mathbf{then}$$
$$\quad\quad x$$
$$\quad \mathbf{else}$$
$$\quad\quad x - \mathrm{floor}(x/y) * y$$
$$\quad \mathbf{end\ if}$$
$$\mathbf{end\ proc}$$

$$5$$
$$0.3615 \qquad (3.3.7.3)$$

▼ *3.3.8. Bővített valós számok.*

▼ *3.3.9. Valós számok kerekítése és fixpontos ábrázolása számítógépben.*

▼ *3.3.10. Valós számok lebegőpontos ábrázolása számítógépben.*

▶ *3.3.11. Tétel: a valós számok létezése.*

▶ **3.3.12. A valós számok más bevezetései.*

▼ *3.3.13. Tétel: gyökvonás.*

▶ *3.3.14. Következmény.*

▼ *3.3.15. A természetes, az egész és a racionális számok bevezetése a valós számok segítségével.*

▶ *->3.2.16. Feladat.*

▶ *->3.2.17. Feladat.*

▶ *3.2.18. Feladat.*

▶ *3.2.19. Feladat.*

▶ *3.2.20. Feladat.*

▶ *3.2.21. Feladat.*
▶ *->3.2.22. Feladat.*
▶ *3.2.23. Feladat.*
▶ *3.2.24. Feladat.*
▶ *3.2.25. Feladat.*
▶ *->3.2.26. Feladat.*
▼ *->3.2.27. Feladat.*
▶ *->3.2.28. Feladat.*
▶ *->3.2.29. Feladat.*
▶ *->3.2.30. Feladat.*
▶ *3.2.31. Feladat.*
▶ *3.2.32. Feladat.*
▶ *3.2.33. Feladat.*
▼ *3.2.34. Feladat: öröknaptár.*
▶ *3.2.35. Feladat.*
▶ *->3.2.36. Feladat.*
▶ *->3.2.37. Feladat.*
▶ *3.2.38. Feladat.*
▶ *3.2.39. Feladat.*
▶ *3.2.40. Feladat.*
▶ *3.2.41. Feladat.*
▶ *3.2.42. Feladat.*
▶ *3.2.43. Feladat.*
▶ *3.2.44. Feladat.*
▶ *3.2.45. Feladat.*
▶ *3.2.46. Feladat.*
▶ *3.2.47. Feladat.*
▶ **3.2.48. Feladat.*
▶ *3.2.49. Feladat.*
▶ **3.2.50. Feladat.*
▶ *3.3.51. További feladatok.*

▼ 3.4. Komplex számok

```
> restart;
```

3.4.1. Komplex számok.

```
> `&+`:=proc(z,w) [z[1]+w[1],z[2]+w[2]] end;
  `&*`:=proc(z,w) [z[1]*w[1]-z[2]*w[2],z[1]*w[2]+z[2]*w[1]]
  end;

  [x,y]&+[0,0]; [x,y]&+[-x,-y]; [x,y]&*[1,0];

  [x,y]&*[x/(x^2+y^2),-y/(x^2+y^2)]; simplify(%);

  [0,1]&*[0,1];
```

$$\&+ := \mathbf{proc}(z, w)\ [z[1] + w[1], z[2] + w[2]]\ \mathbf{end\ proc}$$

$$\&* := \mathbf{proc}(z, w)\ [z[1] * w[1] - z[2] * w[2], z[1] * w[2] + z[2] * w[1]]\ \mathbf{end\ proc}$$

$$[x, y]$$
$$[0, 0]$$
$$[x, y]$$
$$\left[\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}, 0\right]$$
$$[1, 0]$$
$$[-1, 0] \qquad (3.4.1.1)$$

```
> Complex(3,5); z:=3+5*I; w:=-2-6*I; z*w; Re(z); Im(z);
  conjugate(z);
```

$$3 + 5\,\mathrm{I}$$
$$z := 3 + 5\,\mathrm{I}$$
$$w := -2 - 6\,\mathrm{I}$$
$$24 - 28\,\mathrm{I}$$
$$3$$
$$5$$
$$3 - 5\,\mathrm{I} \qquad (3.4.1.2)$$

```
> z:='z';w:='w'; conjugate(z);

  conjugate(conjugate(z));conjugate(z+w);expand(%);conjugate
  (1/z);
```

$$z := z$$
$$w := w$$

$$\overline{z}$$

$$\frac{z}{z+w}$$

$$\frac{\overline{z}+\overline{w}}{\frac{1}{\overline{z}}} \qquad (3.4.1.3)$$

3.4.2. Példa.

```
> 64/(3^(1/2)+I); evalc(%);
```

$$\frac{64}{\sqrt{3}+\mathrm{I}}$$

$$16\sqrt{3}-16\,\mathrm{I} \qquad (3.4.2.1)$$

3.4.3. Komplex szám abszolút értéke.

```
> z:=x+I*y;abs(z);evalc(%);evalc(1/(x+I*y));evalc(conjugate
  (z)/abs(z)^2);
```

$$z := x+\mathrm{I}\,y$$

$$|x+\mathrm{I}\,y|$$

$$\sqrt{x^2+y^2}$$

$$\frac{x}{x^2+y^2}-\frac{\mathrm{I}\,y}{x^2+y^2}$$

$$\frac{x}{x^2+y^2}-\frac{\mathrm{I}\,y}{x^2+y^2} \qquad (3.4.3.1)$$

```
> signum(3+4*I); signum(-5); signum(0);
```

$$\frac{3}{5}+\frac{4}{5}\,\mathrm{I}$$

$$-1$$

$$0 \qquad (3.4.3.2)$$

->3.4.4. Feladat.

->3.4.5. Feladat.

->3.4.6. Feladat.

```
> evalc(2/(1-I)/(3+I)); evalc(1/(3+4*I)^2); evalc((2+I)/I/(
  -3+4*I));
  evalc((3^(1/2)+I)/(1-I)/(3^(1/2)-I)); simplify(%);
  evalc(1/I/(3-2*I)/(1+I)); evalc(1/(1-I)/(1-2*I)/(1+2*I));
```

$$\frac{2}{5}+\frac{1}{5}\,\mathrm{I}$$

$$-\frac{7}{625}-\frac{24}{625}\,\mathrm{I}$$

$$-\frac{11}{25}+\frac{2}{25}\,\mathrm{I}$$

$$\frac{1}{4}\left(\frac{1}{2}\sqrt{3}-\frac{1}{2}\right)\sqrt{3}-\frac{1}{8}\sqrt{3}-\frac{1}{8}+\mathrm{I}\left(\frac{1}{4}\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\right)\sqrt{3}+\frac{1}{8}\sqrt{3}-\frac{1}{8}\right)$$

$$-\frac{1}{4}\sqrt{3}+\frac{1}{4}+\frac{1}{4}\,\mathrm{I}+\frac{1}{4}\,\mathrm{I}\sqrt{3}$$

$$-\frac{1}{26}-\frac{5}{26}\,\mathrm{I}$$

$$\frac{1}{10}+\frac{1}{10}\,\mathrm{I} \qquad (3.4.6.1)$$

->3.4.7. Feladat.

3.4.8. Feladat.

3.4.9. Feladat.

3.4.10. Komplex szám argumentuma és trigonometrikus alakja.

```
> polar(x+I*y); op(1,%); op(2,%%); polar(3+4*I); evalc(%);
  argument(3+I*4);
```

$$\mathrm{polar}(|x+\mathrm{I}\,y|,\ \mathrm{argument}(x+\mathrm{I}\,y))$$

$$|x+\mathrm{I}\,y|$$

$$\mathrm{argument}(x+\mathrm{I}\,y)$$

$$\mathrm{polar}\left(5,\ \arctan\left(\frac{4}{3}\right)\right)$$

$$3+4\,\mathrm{I}$$

$$\arctan\left(\frac{4}{3}\right) \qquad (3.4.10.1)$$

3.4.11. Példa.

```
> z:=16*sqrt(3)-I*16; polar(z);
```

$$z := 16\sqrt{3}-16\,\mathrm{I}$$

$$\mathrm{polar}\left(32,\ -\frac{1}{6}\,\pi\right) \qquad (3.4.11.1)$$

3.4.12. Gyökvonás komplex számból.

3.4.13. Példa.

```
> z:='z'; i:='i'; w:=16*sqrt(3)-I*16; solve(z^5=w,z); z1:=w^
  (1/5);

  r:=abs(w); phi:=argument(w);

  r^(1/5)*(cos(phi/5+i*2*Pi/5)+I*sin(phi/5+i*2*Pi/5))$i=0..4;
  evalf(%);

  solve(z^5=1); map(z->evalf(z*z1),[%]);
```

$$z := z$$

$$i := i$$

$$w := 16\sqrt{3} - 16\,\mathrm{I}$$

Warning, solutions may have been lost

$$z1 := \left(16\sqrt{3} - 16\,\mathrm{I}\right)^{1/5}$$

$$r := 32$$

$$\phi := -\frac{1}{6}\pi$$

$$32^{1/5}\left(\sin\left(\frac{7}{15}\pi\right) - \mathrm{I}\cos\left(\frac{7}{15}\pi\right)\right),$$

$$32^{1/5}\left(\sin\left(\frac{2}{15}\pi\right) + \mathrm{I}\cos\left(\frac{2}{15}\pi\right)\right),$$

$$32^{1/5}\left(-\sin\left(\frac{4}{15}\pi\right) + \mathrm{I}\cos\left(\frac{4}{15}\pi\right)\right), 32^{1/5}\left(-\frac{1}{2}\sqrt{3} - \frac{1}{2}\,\mathrm{I}\right),$$

$$32^{1/5}\left(\sin\left(\frac{1}{15}\pi\right) - \mathrm{I}\cos\left(\frac{1}{15}\pi\right)\right)$$

$$1.989043791 - 0.2090569258\,\mathrm{I},\ 0.8134732860 + 1.827090915\,\mathrm{I},$$

$$-1.486289651 + 1.338261212\,\mathrm{I},\ -1.732050808 - 1.000000000\,\mathrm{I},$$

$$0.4158233818 - 1.956295201\,\mathrm{I}$$

Warning, solutions may have been lost

$$1, -\frac{1}{4} + \frac{1}{4}\sqrt{5} + \frac{1}{4}\,\mathrm{I}\sqrt{2}\sqrt{5+\sqrt{5}},$$

$$-\frac{1}{4} - \frac{1}{4}\sqrt{5} + \frac{1}{4}\,\mathrm{I}\sqrt{2}\sqrt{5-\sqrt{5}},$$

$$-\frac{1}{4} - \frac{1}{4}\sqrt{5} - \frac{1}{4}\,\mathrm{I}\sqrt{2}\sqrt{5-\sqrt{5}},$$

$$-\frac{1}{4}+\frac{1}{4}\sqrt{5}-\frac{1}{4}\,I\sqrt{2}\,\sqrt{5+\sqrt{5}}$$

$$\begin{bmatrix}1.989043791-0.2090569265\,I,\ 0.8134732858+1.827090915\,I,\\ -1.486289651+1.338261212\,I,\\ -1.732050807-0.9999999996\,I,\\ 0.4158233815-1.956295201\,I\end{bmatrix} \quad (3.4.13.1)$$

3.4.14. Az algebra alaptétele.

```
> f:=(x-1)^2*(x-2); f:=expand(f); solve(f,x); solve(x^3=1,x);
  r:=[%];
```

$$f:=(x-1)^2\,(x-2)$$

$$f:=x^3-4\,x^2+5\,x-2$$

```
Warning, solutions may have been lost
```

$$2,\ 1,\ 1$$

```
Warning, solutions may have been lost
```

$$1,\ -\frac{1}{2}+\frac{1}{2}\,I\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\,I\sqrt{3}$$

$$r:=\left[1,\ -\frac{1}{2}+\frac{1}{2}\,I\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\,I\sqrt{3}\right] \quad (3.4.14.1)$$

->3.4.15. Feladat.

->3.4.16. Feladat.

->3.4.17. Feladat.

3.4.18. Feladat.

3.4.19. Feladat.

3.4.20. Feladat.

3.4.21. Feladat.

->3.4.22. Feladat.

->3.4.23. Feladat.

->3.4.24. Feladat.

->3.4.25. Feladat.

3.4.26. Feladat.

3.4.27. Feladat.

3.4.28. Kvaterniók.

```
> `&+`:=(p,q)->[p[1]+q[1],p[2]+q[2]];
```

```
`&*`:=(p,q)->[p[1]*q[1]-conjugate(q[2])*p[2],q[2]*p[1]+p[2]
*conjugate(q[1])];
```

$$\&+ := (p, q) \to [p_1 + q_1, p_2 + q_2]$$

$$\&^* := (p, q) \to [p_1\, q_1 - \overline{q_2}\, p_2, p_1\, q_2 + p_2\, \overline{q_1}] \tag{3.4.28.1}$$

```
> p:=[a+I*b,c+I*d]; p&+[0,0]; p&+[-a-I*b,-c-I*d];
```

$$p := [a + \mathrm{I}\, b, c + \mathrm{I}\, d]$$

$$[a + \mathrm{I}\, b, c + \mathrm{I}\, d]$$

$$[0, 0] \tag{3.4.28.2}$$

```
> p&*[1,0]; [1,0]&*p;

  q:=[(a-I*b)/(a^2+b^2+c^2+d^2),(-c-I*d)/(a^2+b^2+c^2+d^2)];
  p&*q;evalc(%);simplify(%); q&*p;evalc(%);simplify(%);
```

$$[a + \mathrm{I}\, b, c + \mathrm{I}\, d]$$

$$[a + \mathrm{I}\, b, c + \mathrm{I}\, d]$$

$$q := \left[\frac{a - \mathrm{I}\, b}{a^2 + b^2 + c^2 + d^2}, \frac{-c - \mathrm{I}\, d}{a^2 + b^2 + c^2 + d^2}\right]$$

$$\left[\frac{(a + \mathrm{I}\, b)\,(a - \mathrm{I}\, b)}{a^2 + b^2 + c^2 + d^2} - \overline{\frac{-c - \mathrm{I}\, d}{a^2 + b^2 + c^2 + d^2}}\,(c + \mathrm{I}\, d),\right.$$
$$\left.\frac{(a + \mathrm{I}\, b)\,(-c - \mathrm{I}\, d)}{a^2 + b^2 + c^2 + d^2} + (c + \mathrm{I}\, d)\,\overline{\frac{a - \mathrm{I}\, b}{a^2 + b^2 + c^2 + d^2}}\right]$$

$$\left[\frac{a^2 + b^2}{a^2 + b^2 + c^2 + d^2} + \frac{c^2}{a^2 + b^2 + c^2 + d^2} + \frac{d^2}{a^2 + b^2 + c^2 + d^2},\right.$$
$$\frac{-a\,c + b\,d}{a^2 + b^2 + c^2 + d^2} + \frac{c\,a}{a^2 + b^2 + c^2 + d^2} - \frac{d\,b}{a^2 + b^2 + c^2 + d^2}$$
$$\left.+ \mathrm{I}\left(\frac{-b\,c - a\,d}{a^2 + b^2 + c^2 + d^2} + \frac{d\,a}{a^2 + b^2 + c^2 + d^2} + \frac{c\,b}{a^2 + b^2 + c^2 + d^2}\right)\right]$$

$$[1, 0]$$

$$\left[\frac{(a + \mathrm{I}\, b)\,(a - \mathrm{I}\, b)}{a^2 + b^2 + c^2 + d^2} - \frac{\overline{c + \mathrm{I}\, d}\,(-c - \mathrm{I}\, d)}{a^2 + b^2 + c^2 + d^2},\right.$$
$$\left.\frac{(a - \mathrm{I}\, b)\,(c + \mathrm{I}\, d)}{a^2 + b^2 + c^2 + d^2} + \frac{(-c - \mathrm{I}\, d)\,\overline{a + \mathrm{I}\, b}}{a^2 + b^2 + c^2 + d^2}\right]$$

$$\left[\frac{a^2 + b^2}{a^2 + b^2 + c^2 + d^2} + \frac{c^2 + d^2}{a^2 + b^2 + c^2 + d^2}, 0\right]$$

$$[1, 0] \tag{3.4.28.3}$$

```
> z:='z';w:='w';z1:='z1';p:=[z,w];p1:=[z1,w1];p2:=[z2,w2];
```

```
p&*(p1&*p2);expand(%);(p&*p1)&*p2;expand(%);
```

$$z := z$$
$$w := w$$
$$z1 := z1$$
$$p := [z, w]$$
$$p1 := [z1, w1]$$
$$p2 := [z2, w2]$$
$$\left[z\left(z1\,z2-\overline{w2}\,w1\right)-\overline{z1\,w2+w1\,\overline{z2}}\,w,\ z\left(z1\,w2+w1\,\overline{z2}\right)+w\,\overline{z1\,z2-\overline{w2}\,w1}\right]$$
$$\left[z\,z1\,z2-z\,\overline{w2}\,w1-w\,\overline{z1\,w2}-w\,z2\,\overline{w1},\ z\,z1\,w2+z\,w1\,\overline{z2}+w\,\overline{z1\,z2}-w\,w2\,\overline{w1}\right]$$
$$\left[\left(z\,z1-\overline{w1}\,w\right)z2-\overline{w2}\left(z\,w1+w\,\overline{z1}\right),\ \left(z\,z1-\overline{w1}\,w\right)w2+\left(z\,w1+w\,\overline{z1}\right)\overline{z2}\right]$$
$$\left[z\,z1\,z2-w\,z2\,\overline{w1}-z\,\overline{w2}\,w1-\overline{w2}\,w\,\overline{z1},\ z\,z1\,w2-w\,w2\,\overline{w1}+z\,w1\,\overline{z2}+\overline{z2}\,w\,\overline{z1}\right] \quad (3.4.28.4)$$

```
> p&*(p1&+p2);expand(%);(p&*p1)&+(p&*p2);
  (p1&+p2)&*p;expand(%);(p1&*p)&+(p2&*p);
```

$$\left[z\left(z1+z2\right)-\overline{w1+w2}\,w,\ z\left(w1+w2\right)+w\,\overline{z1+z2}\right]$$
$$\left[z\,z1+z\,z2-\overline{w1}\,w-w\,\overline{w2},\ z\,w1+z\,w2+w\,\overline{z1}+w\,\overline{z2}\right]$$
$$\left[z\,z1+z\,z2-\overline{w1}\,w-w\,\overline{w2},\ z\,w1+z\,w2+w\,\overline{z1}+w\,\overline{z2}\right]$$
$$\left[z\left(z1+z2\right)-\overline{w}\left(w1+w2\right),\ \left(z1+z2\right)w+\left(w1+w2\right)\overline{z}\right]$$
$$\left[z\,z1+z\,z2-\overline{w}\,w1-\overline{w}\,w2,\ w\,z1+w\,z2+\overline{z}\,w1+\overline{z}\,w2\right]$$
$$\left[z\,z1+z\,z2-\overline{w}\,w1-\overline{w}\,w2,\ w\,z1+w\,z2+\overline{z}\,w1+\overline{z}\,w2\right] \quad (3.4.28.5)$$

```
> j:=[0,1]; j&*j; [z,0]&+([w,0]&*j);
```

$$j := [0, 1]$$
$$[-1, 0]$$
$$[z, w] \quad (3.4.28.6)$$

```
> k:=[0,I]; k&*k; i:=[I,0]; i&*i; [a,0]&+([b,0]&*i)&+([c,0]&*
  j)&+([d,0]&*k);
```

$$k := [0, \mathrm{I}]$$
$$[-1, 0]$$
$$i := [\mathrm{I}, 0]$$
$$[-1, 0]$$
$$[a+\mathrm{I}\,b,\ c+\mathrm{I}\,d] \quad (3.4.28.7)$$

```
> p:=[a+I*b,c+I*d]; evalc([x,0]&*p); evalc(p&*[x,0]);
```

$$p := [a + \mathrm{I}\,b,\ c + \mathrm{I}\,d]$$
$$[x\,a + \mathrm{I}\,x\,b,\ x\,c + \mathrm{I}\,x\,d]$$
$$[x\,a + \mathrm{I}\,x\,b,\ x\,c + \mathrm{I}\,x\,d] \tag{3.4.28.8}$$

```
> j&*[z,0]; [z,0]&*j;
```

$$[0, \bar{z}]$$
$$[0, z] \tag{3.4.28.9}$$

```
> i&*j; j&*k; k&*i; j&*i; k&*j; i&*k;
```

$$[0, \mathrm{I}]$$
$$[\mathrm{I}, 0]$$
$$[0, 1]$$
$$[0, -\mathrm{I}]$$
$$[-\mathrm{I}, 0]$$
$$[0, -1] \tag{3.4.28.10}$$

```
> i:='i'; j:='j'; k:='k';

  C2toR4:=q->evalc(Re(q[1])+Im(q[1])*i+Re(q[2])*j+Im(q[2])*k)
  ; q:=C2toR4(p);
```

$$i := i$$
$$j := j$$
$$k := k$$
$$C2toR4 := q \to evalc\left(\Re\left(q_1\right) + \Im\left(q_1\right) i + \Re\left(q_2\right) j + \Im\left(q_2\right) k\right)$$
$$q := a + b\,i + c\,j + d\,k \tag{3.4.28.11}$$

```
> R4toC2:=q->[q-coeff(q,i)*i-coeff(q,j)*j-coeff(q,k)*k+I*
  coeff(q,i),coeff(q,j)+I*coeff(q,k)]; R4toC2(q);
```

$$R4toC2 := q \to [q - coeff(q, i)\,i - coeff(q, j)\,j - coeff(q, k)\,k + \mathrm{I}\,coeff(q, i),\ coeff(q, j) + \mathrm{I}\,coeff(q, k)]$$
$$[a + \mathrm{I}\,b,\ c + \mathrm{I}\,d] \tag{3.4.28.12}$$

```
> qIm:=q->coeff(q,i)*i+coeff(q,j)*j+coeff(q,k)*k;qRe:=q->q-
  qIm(q); qRe(q); qIm(q);
```

$$qIm := q \to coeff(q, i)\,i + coeff(q, j)\,j + coeff(q, k)\,k$$
$$qRe := q \to q - qIm(q)$$
$$a$$
$$b\,i + c\,j + d\,k \tag{3.4.28.13}$$

```
> qconjugate:=q->qRe(q)-qIm(q); qconjugate(q);
```

$$qconjugate := q \to qRe(q) - qIm(q)$$

$$a - b\,i - c\,j - d\,k \tag{3.4.28.14}$$

```
> q; qconjugate(q); qconjugate(%); q+qconjugate(q); q-
  qconjugate(q);
```

$$a + b\,i + c\,j + d\,k$$
$$a - b\,i - c\,j - d\,k$$
$$a + b\,i + c\,j + d\,k$$
$$2\,a$$
$$2\,b\,i + 2\,c\,j + 2\,d\,k \tag{3.4.28.15}$$

```
> q1:=a1+b1*i+c1*j+d1*k; q2:=a2+b2*i+c2*j+d2*k;

  q1+q2; collect(%,[i,j,k]); `&+`:=(q1,q2)->collect(q1+q2,[i,
  j,k]); q1&+q2;

  `&*`:=proc(q1,q2) local a1,a2,b1,b2,c1,c2,d1,d2;
  a1:=qRe(q1);a2:=qRe(q2);b1:=coeff(q1,i);b2:=coeff(q2,i);
  c1:=coeff(q1,j);c2:=coeff(q2,j);d1:=coeff(q1,k);d2:=coeff
  (q2,k);
  (a1*a2-b1*b2-c1*c2-d1*d2)+(a1*b2+a2*b1+c1*d2-d1*c2)*i+
  (a1*c2+c1*a2+d1*b2-b1*d2)*j+(a1*d2+d1*a2+b1*c2-c1*b2)*k;
  end;

  q1&*q2;

  qconjugate(q1&+q2); qconjugate(q1)&+qconjugate(q2);

  qconjugate(q1&*q2);qconjugate(q2)&*qconjugate(q1); expand
  (%%-%);
```

$$q1 := a1 + b1\,i + c1\,j + d1\,k$$
$$q2 := a2 + b2\,i + c2\,j + d2\,k$$
$$a1 + b1\,i + c1\,j + d1\,k + a2 + b2\,i + c2\,j + d2\,k$$
$$(b1 + b2)\,i + (c2 + c1)\,j + (d2 + d1)\,k + a1 + a2$$
$$\&+ := (q1, q2) \to collect(q1 + q2, [i, j, k])$$
$$(b1 + b2)\,i + (c2 + c1)\,j + (d2 + d1)\,k + a1 + a2$$

$$\&* := \mathbf{proc}(q1, q2)$$
$$\quad \mathbf{local}\ a1, a2, b1, b2, c1, c2, d1, d2;$$
$$\quad a1 := qRe(q1);$$
$$\quad a2 := qRe(q2);$$
$$\quad b1 := coeff(q1, i);$$
$$\quad b2 := coeff(q2, i);$$
$$\quad c1 := coeff(q1, j);$$

```
    c2 := coeff(q2, j);
    d1 := coeff(q1, k);
    d2 := coeff(q2, k);
    a1*a2 - b1*b2 - c1*c2 - d1*d2 + (a1*b2 + a2*b1
    + c1*d2 - d1*c2)*i + (a1*c2 + c1*a2 + d1*b2 - b1*d2)*j
    + (a1*d2 + d1*a2 + b1*c2 - c1*b2)*k
end proc
```

$$a1\,a2 - b1\,b2 - c1\,c2 - d1\,d2 + (a1\,b2 + a2\,b1 + c1\,d2 - d1\,c2)\,i + (a1\,c2 + c1\,a2 + d1\,b2 - b1\,d2)\,j + (a1\,d2 + d1\,a2 + b1\,c2 - c1\,b2)\,k$$

$$a1 + a2 - (b1 + b2)\,i - (c2 + c1)\,j - (d2 + d1)\,k$$

$$(-b1 - b2)\,i + (-c2 - c1)\,j + (-d2 - d1)\,k + a1 + a2$$

$$a1\,a2 - b1\,b2 - c1\,c2 - d1\,d2 - (a1\,b2 + a2\,b1 + c1\,d2 - d1\,c2)\,i - (a1\,c2 + c1\,a2 + d1\,b2 - b1\,d2)\,j - (a1\,d2 + d1\,a2 + b1\,c2 - c1\,b2)\,k$$

$$a1\,a2 - b1\,b2 - c1\,c2 - d1\,d2 + (-a2\,b1 - a1\,b2 + d1\,c2 - c1\,d2)\,i + (-c1\,a2 - a1\,c2 + b1\,d2 - d1\,b2)\,j + (-d1\,a2 - a1\,d2 + c1\,b2 - b1\,c2)\,k$$

$$0 \qquad (3.4.28.16)$$

3.4.29. Kvaterniók abszolút értéke.

```
> qabs:=q->sqrt(qRe(q)^2+coeff(q,i)^2+coeff(q,j)^2+coeff(q,k)
  ^2);

  qabs(q); q&*qconjugate(q);
```

$$qabs := q \to \sqrt{qRe(q)^2 + coeff(q, i)^2 + coeff(q, j)^2 + coeff(q, k)^2}$$

$$\sqrt{a^2 + b^2 + c^2 + d^2}$$

$$a^2 + b^2 + c^2 + d^2 \qquad (3.4.29.1)$$

->3.4.30. Feladat.

*3.4.31. Vektoriális szorzás.

*3.4.32. Kvaterniók és a háromdimenziós euklidészi tér.

```
> q1:=b1*i+c1*j+d1*k; q2:=b2*i+c2*j+d2*k; q3:=b3*i+c3*j+d3*k;

  scalarprod:=(q1,q2)->-qRe(q1&*q2); scalarprod(q1,q2);
```

```
vectorprod:=(q1,q2)->qIm(q1&*q2); vectorprod(q1,q2);

mixedprod:=(q1,q2,q3)->scalarprod(q1,vectorprod(q2,q3));
mixedprod(q1,q2,q3);
```

$$q1 := b1\,i + c1\,j + d1\,k$$
$$q2 := b2\,i + c2\,j + d2\,k$$
$$q3 := b3\,i + c3\,j + d3\,k$$
$$scalarprod := (q1, q2) \rightarrow -qRe(q1\ \&^*\ q2)$$
$$b1\,b2 + c1\,c2 + d1\,d2$$
$$vectorprod := (q1, q2) \rightarrow qIm(q1\ \&^*\ q2)$$
$$(c1\,d2 - d1\,c2)\,i + (d1\,b2 - b1\,d2)\,j + (b1\,c2 - c1\,b2)\,k$$
$$mixedprod := (q1, q2, q3) \rightarrow scalarprod(q1, vectorprod(q2, q3))$$
$$b1\,(c2\,d3 - d2\,c3) + c1\,(d2\,b3 - b2\,d3) + d1\,(b2\,c3 - c2\,b3) \qquad (3.4.32.1)$$

▼ *3.4.33. A szorzások geometriai jelentése.

▼ *3.4.34. Forgatások.

▼ *3.4.35. A skaláris es vektoriális szorzás geometriai alkalmazásai.

▼ * 3.4.36. Oktávok vagy Cayley-számok.

▼ ->3.4.37. Feladat.

▼ ->3.4.38. Feladat.

▼ *3.4.39. Feladat.

▶ 3.4.40. További feladatok megoldásokkal.

▶ 3.4.41. További feladatok.

▶ 4. Véges halmazok

▶ 5. Végtelen halmazok

▶ 6. Számelmélet

▶ 7. Gráfelmélet

▶ 8. Algebra

▶ 9. Kódolás

▶ 10. Algoritmusok