Kalkulus II.

Járai Antal

Ezek a programok csak szemléltetésre szolgálnak

1. Lineáris algebra

2. Többváltozós függvények

2.1. Határérték és folytonosság

```
> restart;
```

2.1.1. Távolság, környezetek, korlátosság.

```
> verify([3,3,3],[Pi,Pi,Pi],'neighborhood(1)');
```

true (2.1.1.1)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1)');
```

false (2.1.1.2)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1,closed)');
```

true (2.1.1.3)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1,p=1)');
```

false (2.1.1.4)

```
> verify([1/sqrt(3),1/sqrt(3),1/sqrt(3)],[0,0,0],
  'neighborhood(1,p=3)');
```

true (2.1.1.5)

2.1.2. Belső, külső, izolált, torlódási és határpontok.

2.1.3. Nyílt és zárt halmazok.

2.1.4. Állítás.

2.1.5. Állítás.

2.1.6. Tétel.

2.1.7. Tétel.

▶ *2.1.8. Következmény.*

▶ *2.1.9. Következmény.*

▶ *2.1.10. Következmény.*

▶ *2.1.11. Tartomány.*

▶ *2.1.12. Konvex halmaz.*

▶ *2.1.13. Sűrű halmazok.*

▶ *2.1.14. Kompakt halmazok.*

▶ *2.1.15. Tétel.*

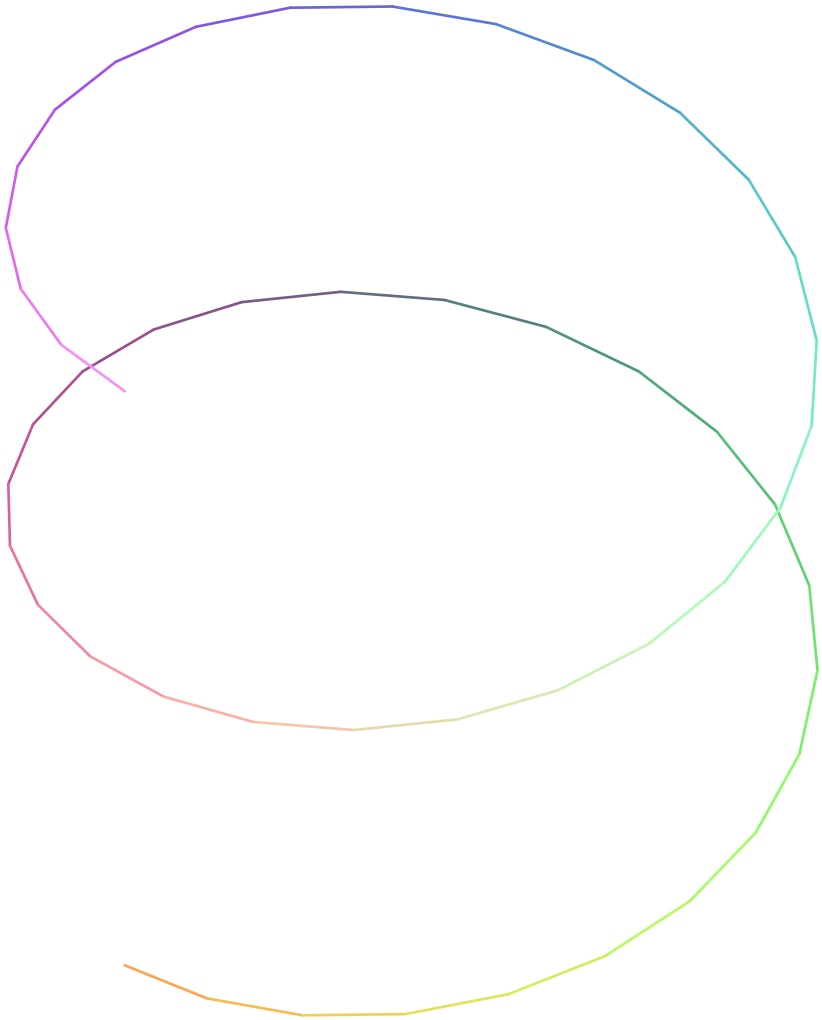
▶ *2.1.16. Következmény: Weierstrass tétele.*

▼ *2.1.17.Vektor-skalár, skalár-vektor és vektor-vektor függvények.*

```
> with(plots);
```

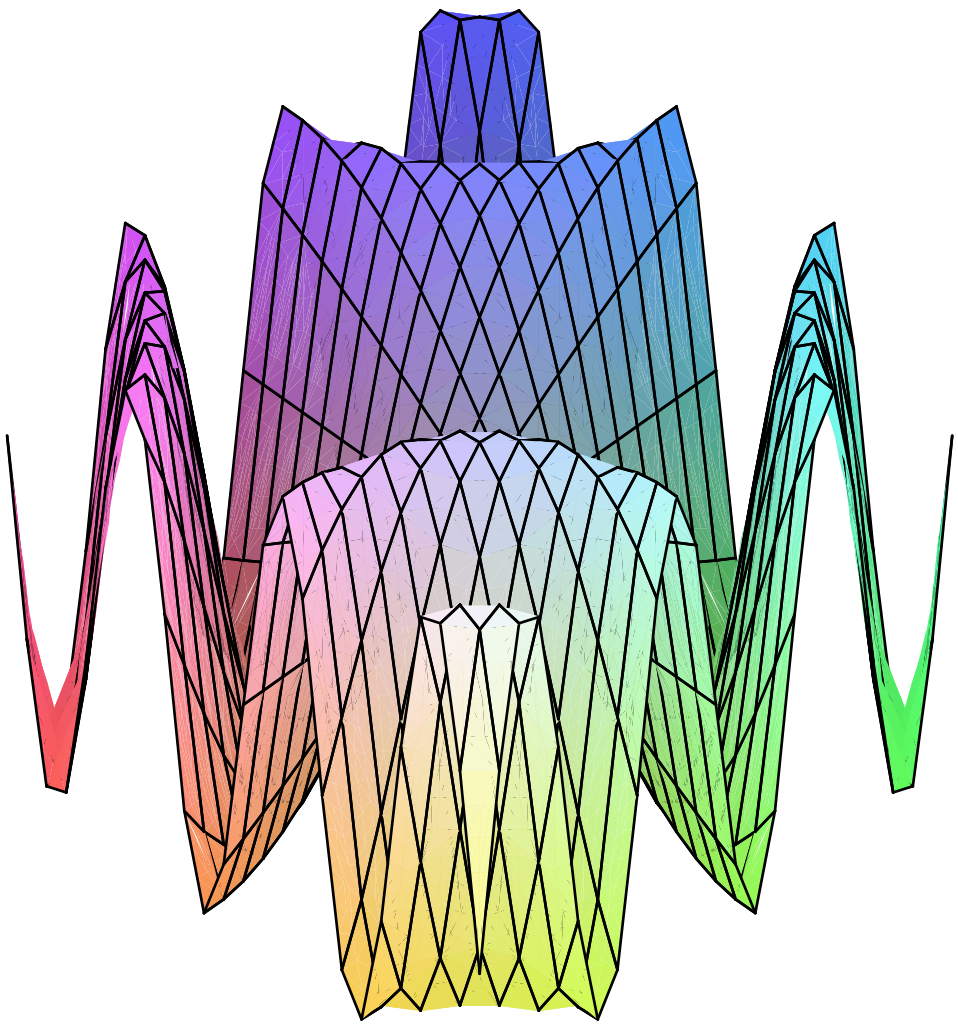
[*Interactive*, *animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *cylinderplot*, *densityplot*, *display*, *display3d*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *graphplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *replot*, *rootlocus*, *semilogplot*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *sphereplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*] (2.1.17.1)

```
> spacecurve([cos(t),sin(t),t],t=0..4*Pi);
```

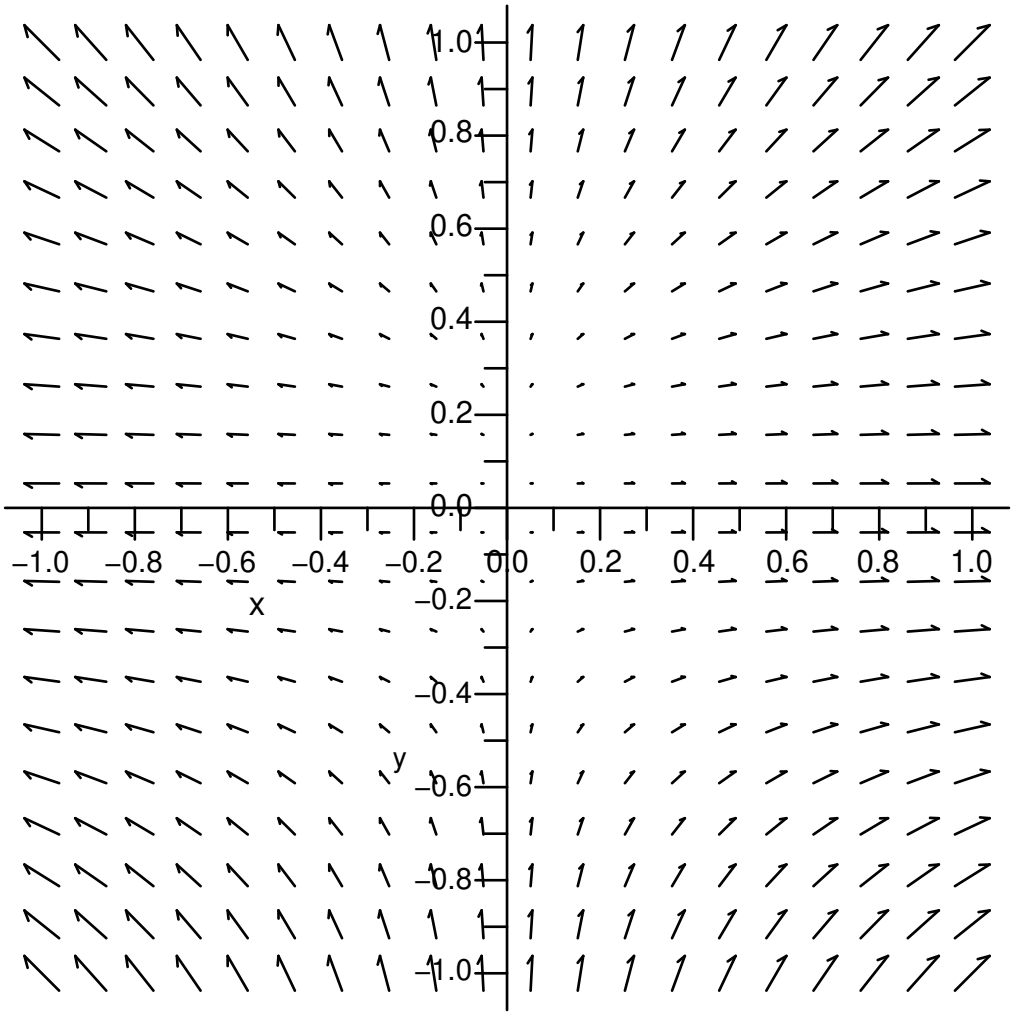


```
> p:=proc(x,y) sin(x*y) end; plot3d(p,-Pi..Pi,-Pi..Pi);
```

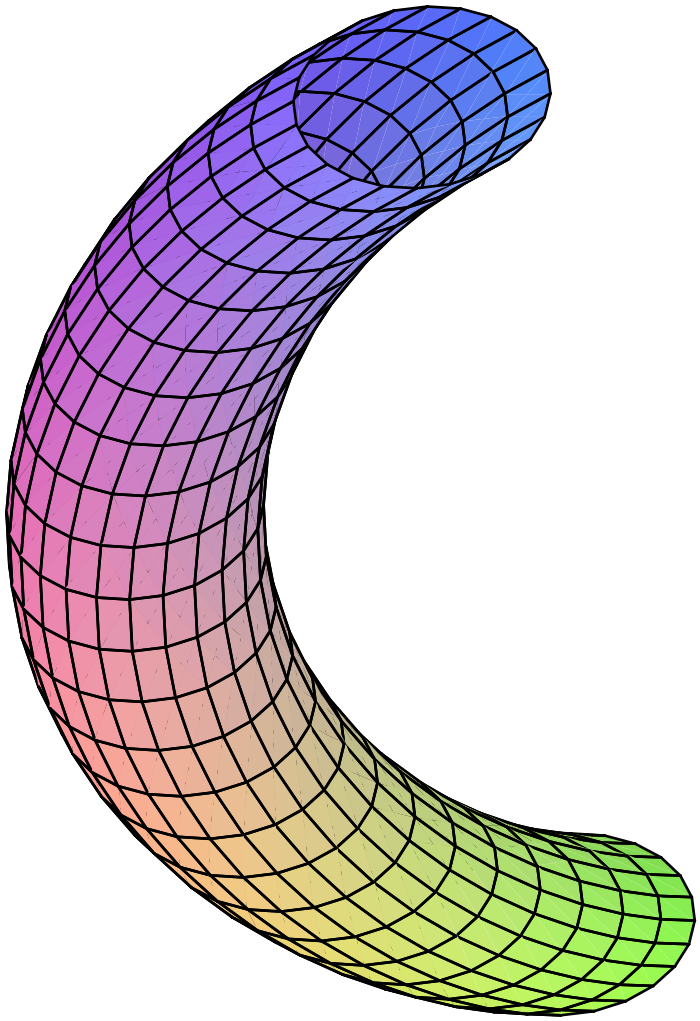
$$p := \mathbf{proc}(x, y)\ \sin(y^{*}x)\ \mathbf{end\ proc}$$



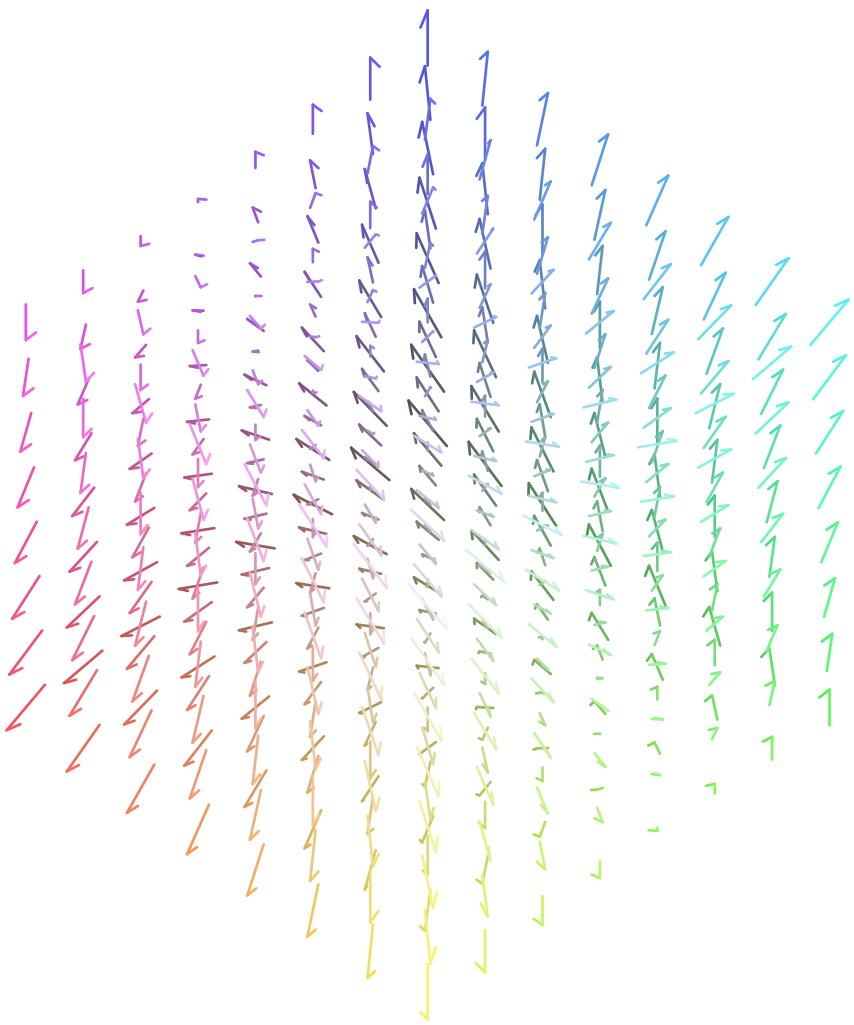
```
> fieldplot([x,y^2],x=-1..1,y=-1..1);
```




```
> plot3d([sin(x)+2*sin(0.4*y),cos(x)+2*cos(0.4*y),y],x=0..2*
  Pi,y=0..10);
```

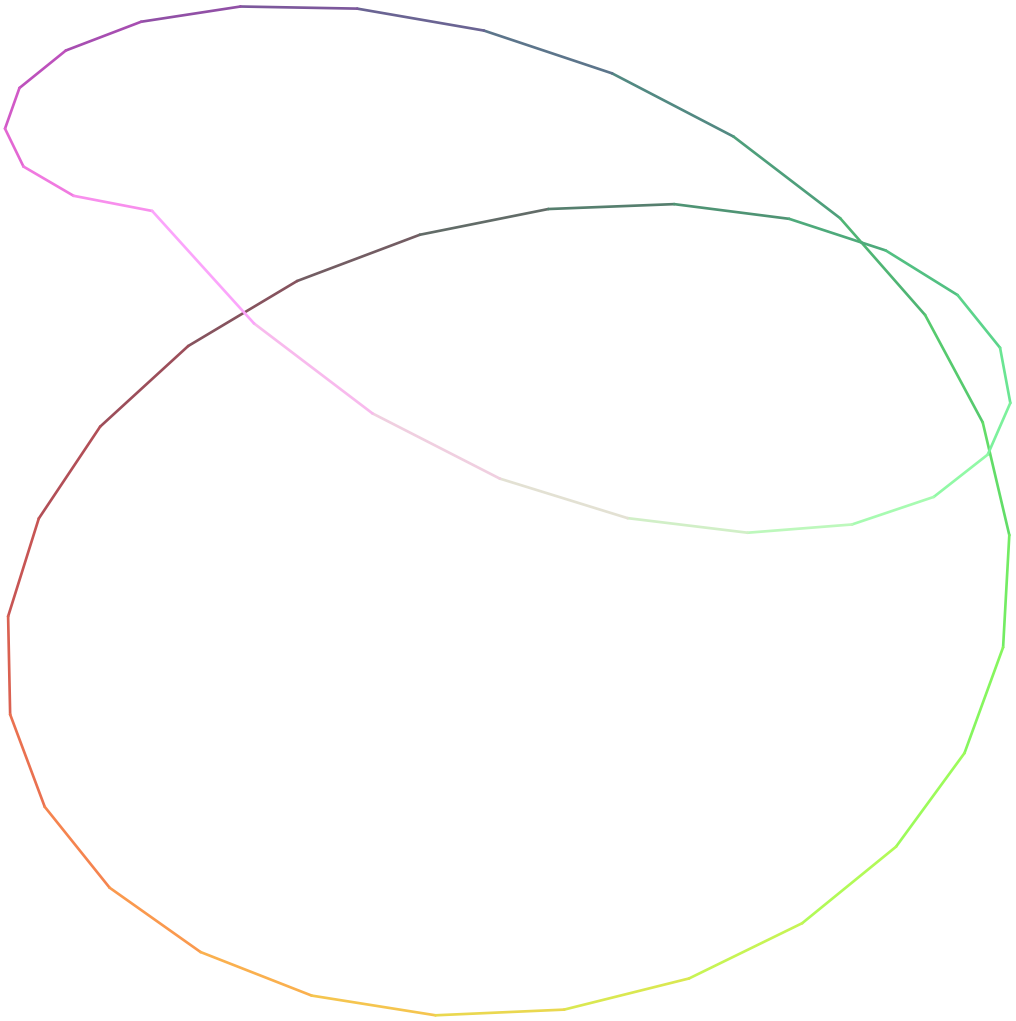


```
> fieldplot3d([x-y,x+z,z-y-x],x=-1..1,y=-1..1,z=-1..1);
```

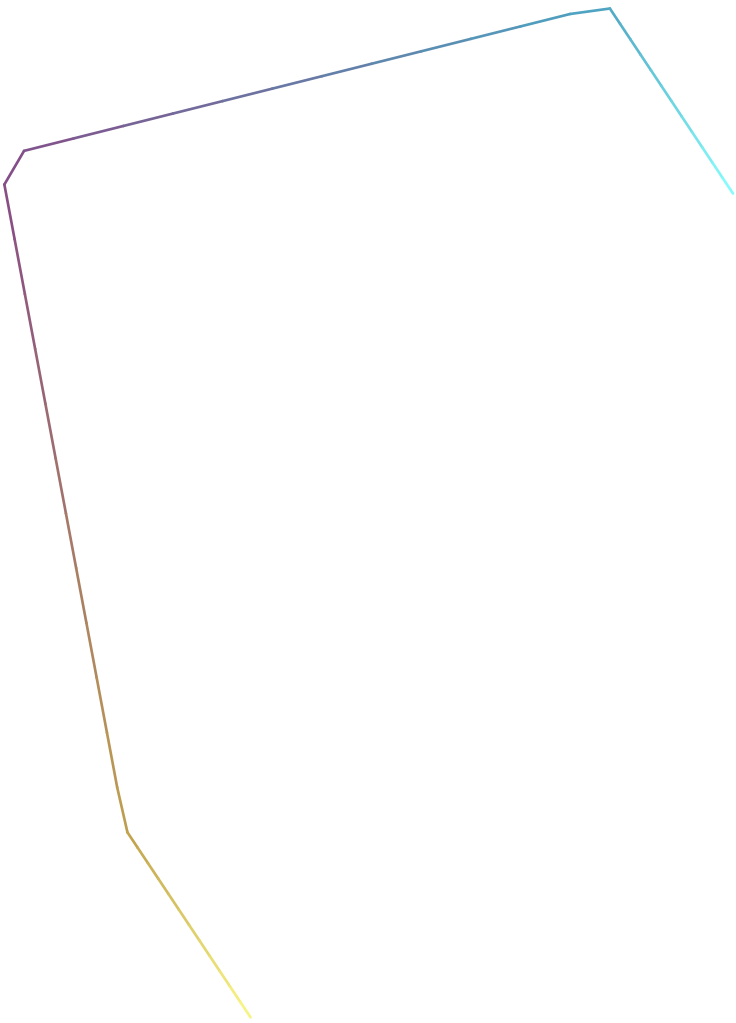


▶ *2.1.18. Folytonosság.*
▶ *2.1.19. Példák.*
▶ *2.1.20. Tétel.*
▶ *2.1.21. Segédtétel.*
▶ *2.1.22. Tétel.*
▶ *2.1.23. Tétel.*
▶ *2.1.24. Példák.*
▼ *2.1.25. Görbék.*

```
> spacecurve([cos(t),sin(t),t*(t-4*Pi)],t=0..4*Pi);
```

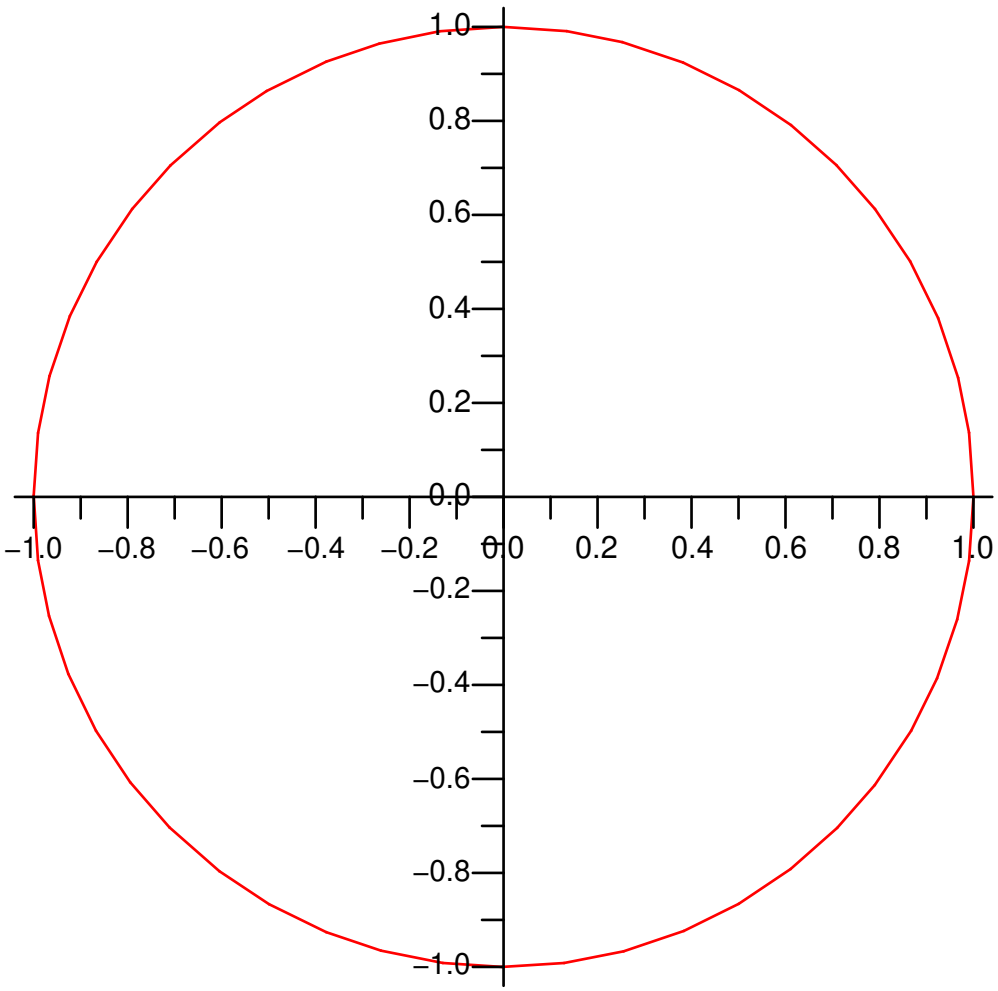


```
> spacecurve([abs(t-1),3+abs(t-2),4+abs(t-3)],t=0..4);
```



- *2.1.26. Tétel.
- *2.1.27. Következmény.
- 2.1.28. Tétel.
- 2.1.29. Következmény: Weierstrass tétele.
- 2.1.30. Hausdorff tétele.
- o 2.1.31. Példa.

```
> plot([cos(t),sin(t),t=0..2*Pi]);
```



▶ *2.1.32. Lebesgue-szám.

▶ *2.1.33. Tétel.

▶ *2.1.34. Egyenletes folytonosság.

▶ *2.1.35. Heine tétele.

▶ 2.1.36. Jobb és bal oldali folytonosság.

▼ 2.1.37. Határérték.

```
> limit((x^4-y^4)/(x^2+y^2),{x=0,y=0});
```

$$0 \qquad (2.1.37.1)$$

```
> limit((x^2-y^2)/(x^2+y^2),{x=0,y=0});
```

$$\mathit{undefined} \qquad (2.1.37.2)$$

```
> limit((sin(x^2)-sin(y^2))/(x-y), {x=0,y=0});
```

$$\mathit{limit}\left(\frac{\sin(x^2)-\sin(y^2)}{x-y}, \{x=0, y=0\}\right) \qquad (2.1.37.3)$$

▶ *2.1.38. Tétel.*
▶ *2.1.39. Tétel.*
▶ *2.1.40. Jobb és bal oldali határérték.*
▶ *2.1.41. Szakadások.*
▶ *2.1.42. Végtelen, mint határérték, határérték a végtelenben.*
▶ *2.1.43. Tétel.*
▶ *2.1.44. Tétel.*
▶ *2.1.45. Tétel: rendőr-elv.*
▶ *2.1.46. Sorozatok.*
▶ *2.1.47. Tétel.*
▶ *2.1.48. Tétel.*
▶ *2.1.49. Tétel.*
▶ *2.1.50. Következmény: Bolzano-Weierstrass-féle kiválasztási tétel.*
▶ *2.1.51. Cauchy-sorozatok.*
▶ *2.1.52. Cauchy-féle konvergenciakritérium.*
▶ *2.1.53. Banach-terek és Hilbert-terek.*
▼ *2.1.54. Sorok.*

```
> sum('1/k!','k'=0..infinity);
```

$$e \tag{2.1.54.1}$$

```
> sum('1/k^2','k'=1..infinity);
```

$$\frac{1}{6}\pi^2 \tag{2.1.54.2}$$

```
> sum('1/k^3','k'=1..infinity);
```

$$\zeta(3) \tag{2.1.54.3}$$

```
> ?Zeta
```

▶ *2.1.55. Tétel.*
▶ *2.1.56. Cauchy-féle konvergenciakritérium.*
▶ *2.1.57. Következmény.*
▶ *2.1.58. Öszehasonlítókritérium.*
▶ *2.1.59. Következmény.*
▶ *2.1.60. Cauchy-féle gyökkritérium.*
▶ *2.1.61. d'Alembert-féle hányadoskritérium.*

▶ *2.1.62. Kettős sor tétel.*
▶ *2.1.63. Következmény: sorok átrendezése.*
▶ *2.1.64. Következmény.*
▶ *2.1.65. Ekvivalens normák.*
▶ *2.1.66. Tétel.*
▶ **2.1.67. Fixpont és kontrakció.*
▶ **2.1.68 Banach–féle fixponttétel.*
▶ **2.1.69. Metrikus terek.*

▼ 2.2. Differenciálszámítás

```
> restart; with(LinearAlgebra); with(VectorCalculus);
```

[*&x*, *Add*, *Adjoint*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BezoutMatrix*, *BidiagonalForm*, *BilinearForm*, *CharacteristicMatrix*, *CharacteristicPolynomial*, *Column*, *ColumnDimension*, *ColumnOperation*, *ColumnSpace*, *CompanionMatrix*, *ConditionNumber*, *ConstantMatrix*, *ConstantVector*, *Copy*, *CreatePermutation*, *CrossProduct*, *DeleteColumn*, *DeleteRow*, *Determinant*, *Diagonal*, *DiagonalMatrix*, *Dimension*, *Dimensions*, *DotProduct*, *EigenConditionNumbers*, *Eigenvalues*, *Eigenvectors*, *Equal*, *ForwardSubstitute*, *FrobeniusForm*, *GaussianElimination*, *GenerateEquations*, *GenerateMatrix*, *GetResultDataType*, *GetResultShape*, *GivensRotationMatrix*, *GramSchmidt*, *HankelMatrix*, *HermiteForm*, *HermitianTranspose*, *HessenbergForm*, *HilbertMatrix*, *HouseholderMatrix*, *IdentityMatrix*, *IntersectionBasis*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*, *JordanBlockMatrix*, *JordanForm*, *LA_Main*, *LUDecomposition*, *LeastSquares*, *LinearSolve*, *Map*, *Map2*, *MatrixAdd*, *MatrixExponential*, *MatrixFunction*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*,

ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

[*&x, *, +, -, ., <,>, </>, AddCoordinates, ArcLength, BasisFormat, Binormal, CrossProd, CrossProduct, Curl, Curvature,* D, *Del, DirectionalDiff, Divergence, DotProd, DotProduct, Flux, GetCoordinateParameters, GetCoordinates, Gradient, Hessian, Jacobian, Laplacian, LineInt, MapToBasis, Nabla, Norm, Normalize, PathInt, PrincipalNormal, RadiusOfCurvature, ScalarPotential, SetCoordinateParameters, SetCoordinates, SurfaceInt, TNBFrame, Tangent, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, Wronskian, diff, evalVF, int, limit, series*] (2.2.1)

2.2.1. Lineáris operátorok normája.

```
> a:=Matrix([[1,0],[2,3]]);
```

$$a := \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \tag{2.2.1.1}$$

```
> MatrixNorm(a); MatrixNorm(a,'infinity');
  MatrixNorm(a,1); MatrixNorm(a,2); evalf(%); MatrixNorm(a,
  Frobenius);
```

$$5$$

$$5$$

$$3$$

$$\sqrt{\text{RootOf}\left(_Z^2 - 14\,_Z + 9,\ index = 2\right)}$$

$$3.650281540$$

$$\sqrt{14} \tag{2.2.1.2}$$

2.2.2. Feladat.

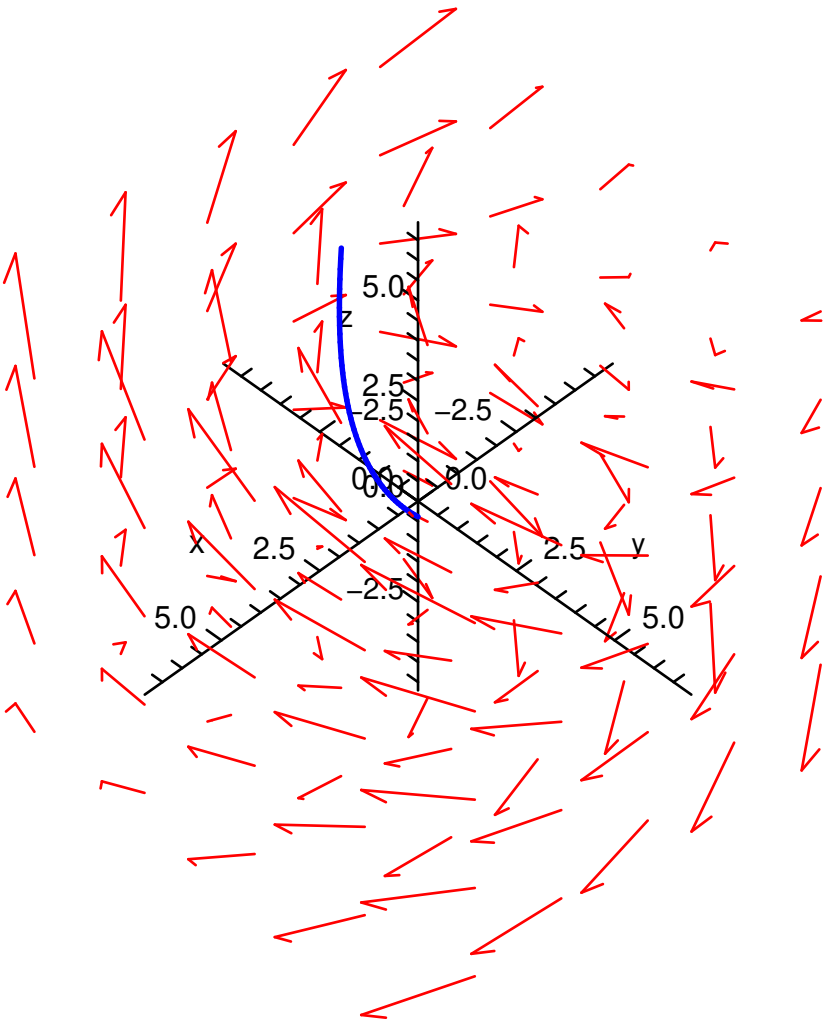
2.2.3. Feladat.

▶ *2.2.4. Feladat.*

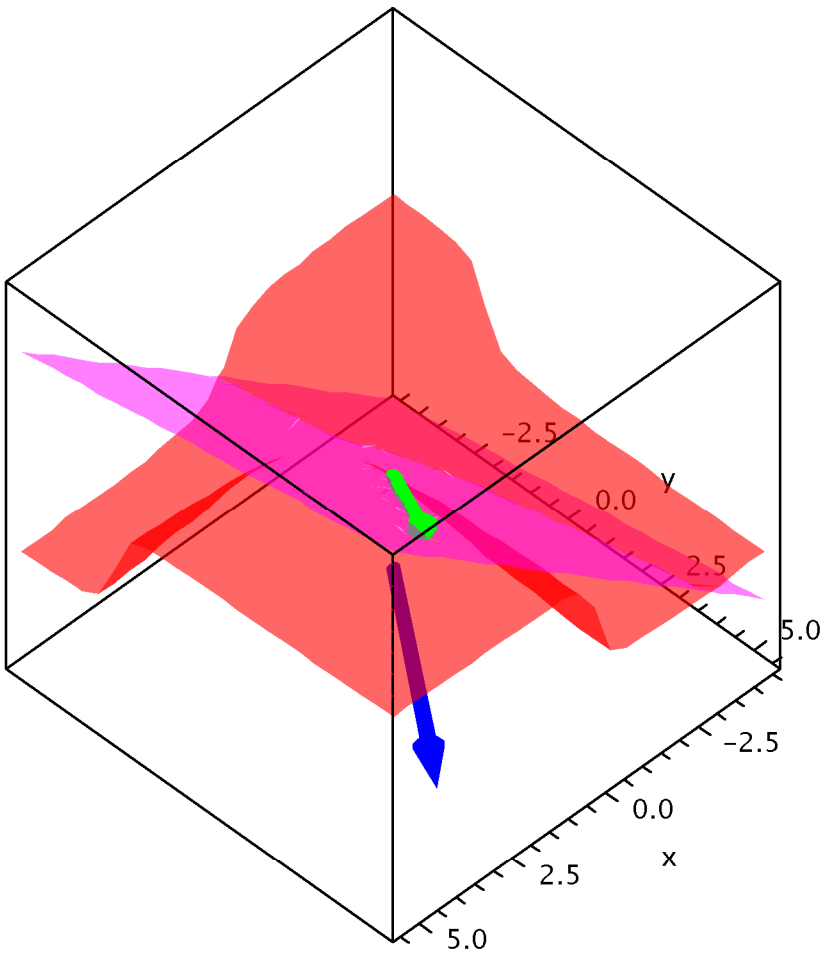
▶ *2.2.5. Feladat.*

▼ *2.2.6. Definíció.*

```
> Student[VectorCalculus][VectorFieldTutor]();
```




```
> Student[MultivariateCalculus][DirectionalDerivativeTutor]()
  ;
```




```
> f:=<x^2,x*y,x*z>;
  Jacobian(f,[x,y,z]);
  diff(f[1],x); diff(f[1],y); diff(f[1],z);
  diff(f[2],x); diff(f[2],y); diff(f[2],z);
  diff(f[3],x); diff(f[3],y); diff(f[3],z);
```

$$f := (x^2)e_x + (xy)e_y + (xz)e_z$$

$$\begin{bmatrix} 2x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \end{bmatrix}$$

$$2x$$

$$0$$

$$0$$

$$y$$

$$x$$

$$\begin{matrix} 0 \\ z \\ 0 \\ x \end{matrix} \tag{2.2.6.1}$$

```
> f:=<x^2,x*y,x*z,z*y>; Jacobian(f,[x,y,z]);
```

$$f := \left(x^2\right) e_{x1} + (x\,y) e_{x2} + (x\,z) e_{x3} + (z\,y) e_{x4}$$

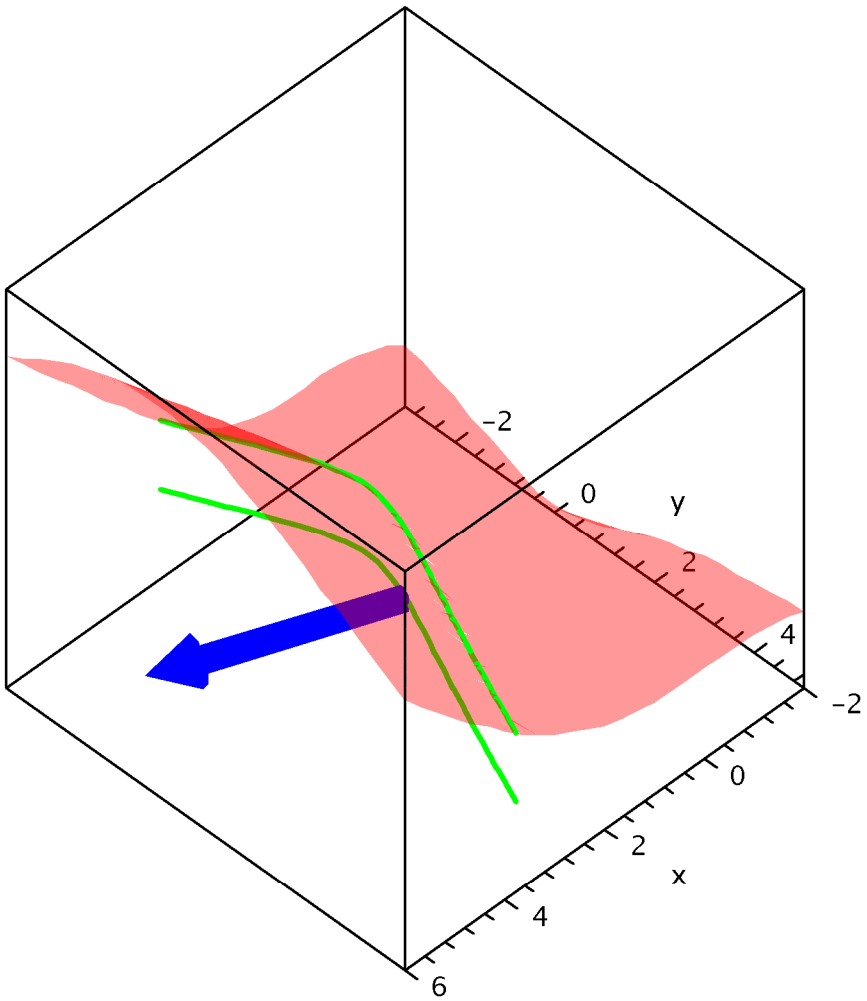
$$\begin{bmatrix} 2\,x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix} \tag{2.2.6.2}$$

```
> f:=3*x^2+2*y*z; Gradient(f,[x,y,z]);
```

$$f := 3\,x^2 + 2\,z\,y$$

$$6\,x\bar{e}_x + 2\,z\bar{e}_y + 2\,y\bar{e}_z \tag{2.2.6.3}$$

```
> Student[MultivariateCalculus][GradientTutor]();
```



- *2.2.7. Sima görbék.*
- *2.2.8. Összegszabály.*
- *2.2.9. Láncszabály.*
- *2.2.10. Koordinátafüggvények.*
- *2.2.11. Parciális<deriváltak.*
- *2.2.12. Középérték egyenlőtlenség.*
- *2.2.13. Következmény.*
- *2.2.14. Következmény.*
- *2.2.15. Tétel.*
- *2.2.16. Magasabbrendű deriváltak.*

```
> f:=(x,y,z)->x^2*y*z^3;
  D[1](D[1](f)); D[1](D[2](f)); D[2](D[1](f));
```

$$f := (x, y, z) \to \mathit{VectorCalculus}:-*\left(\mathit{VectorCalculus}:-*\left(x^2, y\right), z^3\right)$$

$$(x, y, z) \to 2\,y\,z^3$$

$$(x, y, z) \to 2\,x\,z^3$$

$$(x, y, z) \to 2\,x\,z^3 \qquad (2.2.16.1)$$

```
> f:=<3*x^3+2*y*z*x+x^2*y^2*z^2>; Jacobian(f,[x,y,z]);
  convert(%,list); Jacobian(%,[x,y,z]);
  convert(%,listlist); map(u->Jacobian(u,[x,y,z]),%);
```

$$f := \left(3\,x^3 + 2\,z\,y\,x + x^2\,y^2\,z^2\right) e_x$$

$$\left[9\,x^2 + 2\,z\,y + 2\,x\,y^2\,z^2 \quad 2\,x\,z + 2\,x^2\,y\,z^2 \quad 2\,x\,y + 2\,x^2\,y^2\,z\right]$$

$$\left[9\,x^2 + 2\,z\,y + 2\,x\,y^2\,z^2,\ 2\,x\,z + 2\,x^2\,y\,z^2,\ 2\,x\,y + 2\,x^2\,y^2\,z\right]$$

$$\begin{bmatrix} 18\,x + 2\,y^2\,z^2 & 2\,z + 4\,x\,y\,z^2 & 2\,y + 4\,x\,y^2\,z \\ 2\,z + 4\,x\,y\,z^2 & 2\,x^2\,z^2 & 2\,x + 4\,x^2\,y\,z \\ 2\,y + 4\,x\,y^2\,z & 2\,x + 4\,x^2\,y\,z & 2\,x^2\,y^2 \end{bmatrix}$$

$$\left[\left[18\,x + 2\,y^2\,z^2,\ 2\,z + 4\,x\,y\,z^2,\ 2\,y + 4\,x\,y^2\,z\right], \left[2\,z + 4\,x\,y\,z^2,\ 2\,x^2\,z^2,\right.\right.$$
$$\left.\left.2\,x + 4\,x^2\,y\,z\right], \left[2\,y + 4\,x\,y^2\,z,\ 2\,x + 4\,x^2\,y\,z,\ 2\,x^2\,y^2\right]\right]$$

$$\left[\begin{bmatrix} 18 & 4\,y\,z^2 & 4\,y^2\,z \\ 4\,y\,z^2 & 4\,x\,z^2 & 2 + 8\,z\,y\,x \\ 4\,y^2\,z & 2 + 8\,z\,y\,x & 4\,x\,y^2 \end{bmatrix}, \begin{bmatrix} 4\,y\,z^2 & 4\,x\,z^2 & 2 + 8\,z\,y\,x \\ 4\,x\,z^2 & 0 & 4\,x^2\,z \\ 2 + 8\,z\,y\,x & 4\,x^2\,z & 4\,x^2\,y \end{bmatrix}, \right.$$
$$\left.\begin{bmatrix} 4\,y^2\,z & 2 + 8\,z\,y\,x & 4\,x\,y^2 \\ 2 + 8\,z\,y\,x & 4\,x^2\,z & 4\,x^2\,y \\ 4\,x\,y^2 & 4\,x^2\,y & 0 \end{bmatrix}\right] \qquad (2.2.16.2)$$

```
> f:=<x^3,x*y*z,x*z^2,z*y^2>; Jacobian(f,[x,y,z]);
  convert(%,listlist); map(u->Jacobian(u,[x,y,z]),%);
```

$$f := \left(x^3\right) e_{x1} + (z\,y\,x)\,e_{x2} + \left(x\,z^2\right) e_{x3} + \left(y^2\,z\right) e_{x4}$$

$$\begin{bmatrix} 3\,x^2 & 0 & 0 \\ z\,y & x\,z & x\,y \\ z^2 & 0 & 2\,x\,z \\ 0 & 2\,z\,y & y^2 \end{bmatrix}$$

$$\left[\left[3\,x^2, 0, 0\right], \left[z\,y, x\,z, x\,y\right], \left[z^2, 0, 2\,x\,z\right], \left[0, 2\,z\,y, y^2\right]\right]$$

$$\left[\begin{bmatrix} 6x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2z \\ 0 & 0 & 0 \\ 2z & 0 & 2x \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{bmatrix}\right] \quad (2.2.16.3)$$

▶ *2.2.17. Magasabbrendű deriváltak és parciális deriváltak kapcsolata.*

▼ *2.2.18. A második derivált mint bilineáris leképezés.*

```
> Hessian(x^2*y*z^3,[x,y,z]);
```

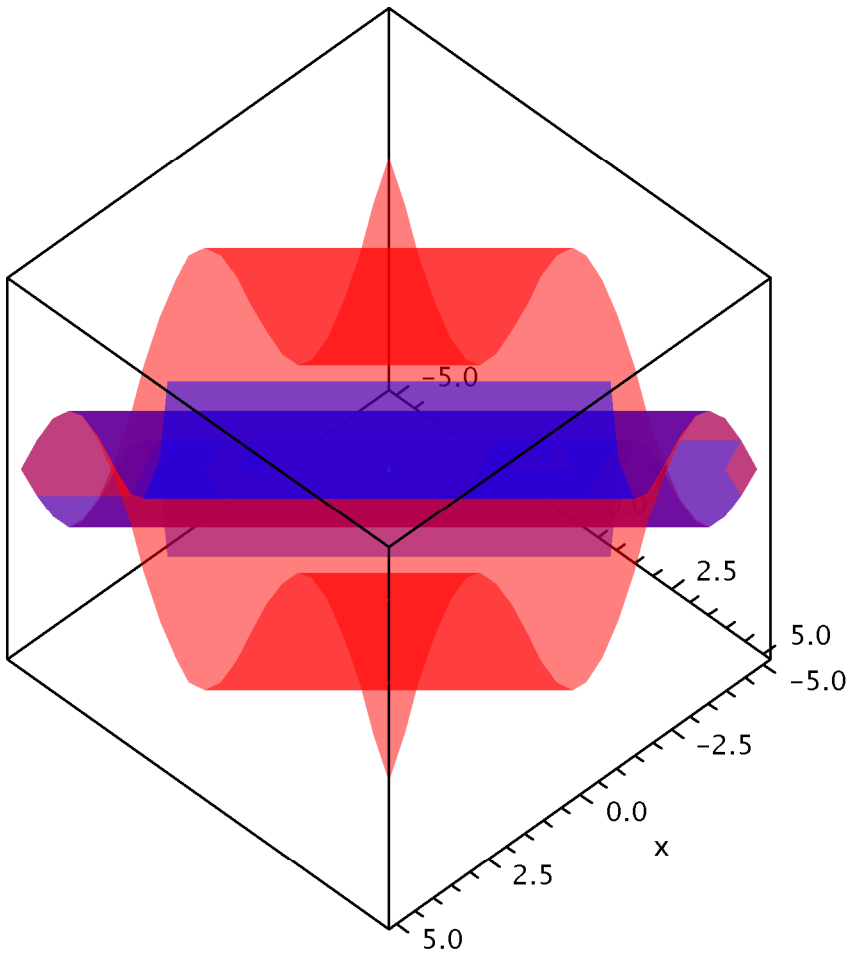
$$\begin{bmatrix} 2yz^3 & 2xz^3 & 6xyz^2 \\ 2xz^3 & 0 & 3x^2z^2 \\ 6xyz^2 & 3x^2z^2 & 6x^2yz \end{bmatrix} \quad (2.2.18.1)$$

▶ **2.2.19. Magasabbrendű deriváltak mint multilineáris leképezések.*

▶ *2.2.20. Young tétele.*

▼ *2.2.21. Taylor-formula valós értékű függvényekre.*

```
> Student[MultivariateCalculus][TaylorApproximationTutor]();
```




```
> mtaylor(sin(x^2+y^2),[x,y]);
  mtaylor(sin(x^2+y^2),[x=0,y=0]);
  mtaylor(sin(x^2+y^2),[x,y],8);
  mtaylor(sin(x^2+y^2),[x=1,y=2],3);
  mtaylor(g(x,y),[x,y]);
```

$$x^2+y^2$$

$$x^2+y^2$$

$$x^2+y^2-\frac{1}{6}x^6-\frac{1}{2}y^2x^4-\frac{1}{2}y^4x^2-\frac{1}{6}y^6$$

$$\sin(5)+2\cos(5)(x-1)+4\cos(5)(y-2)+(-2\sin(5)+\cos(5))(x-1)^2-8\sin(5)(y-2)(x-1)+(-8\sin(5)+\cos(5))(y-2)^2$$

$$g(0,0)+D_1(g)(0,0)x+D_2(g)(0,0)y+\frac{1}{2}D_{1,1}(g)(0,$$ (2.2.21.1)

$$0)\,x^2 + x\,\mathrm{D}_{1,\,2}(g)(0,\,0)\,y + \frac{1}{2}\,\mathrm{D}_{2,\,2}(g)(0,$$

$$0)\,y^2 + \frac{1}{6}\,x^3\,\mathrm{D}_{1,\,1,\,1}(g)(0,\,0) + \frac{1}{2}\,x^2\,\mathrm{D}_{1,\,1,\,2}(g)(0,$$

$$0)\,y + \frac{1}{2}\,x\,y^2\,\mathrm{D}_{1,\,2,\,2}(g)(0,\,0) + \frac{1}{6}\,y^3\,\mathrm{D}_{2,\,2,\,2}(g)(0,$$

$$0) + \frac{1}{24}\,x^4\,\mathrm{D}_{1,\,1,\,1,\,1}(g)(0,\,0) + \frac{1}{6}\,x^3\,\mathrm{D}_{1,\,1,\,1,\,2}(g)(0,$$

$$0)\,y + \frac{1}{4}\,x^2\,y^2\,\mathrm{D}_{1,\,1,\,2,\,2}(g)(0,\,0) + \frac{1}{6}\,x\,y^3\,\mathrm{D}_{1,\,2,\,2,\,2}(g)(0,$$

$$0) + \frac{1}{24}\,y^4\,\mathrm{D}_{2,\,2,\,2,\,2}(g)(0,\,0) + \frac{1}{120}\,x^5\,\mathrm{D}_{1,\,1,\,1,\,1,\,1}(g)(0,$$

$$0) + \frac{1}{24}\,x^4\,\mathrm{D}_{1,\,1,\,1,\,1,\,2}(g)(0,\,0)\,y + \frac{1}{12}\,x^3\,y^2\,\mathrm{D}_{1,\,1,\,1,\,2,\,2}(g)(0,$$

$$0) + \frac{1}{12}\,x^2\,y^3\,\mathrm{D}_{1,\,1,\,2,\,2,\,2}(g)(0,\,0) + \frac{1}{24}\,x\,y^4\,\mathrm{D}_{1,\,2,\,2,\,2,\,2}(g)(0,$$

$$0) + \frac{1}{120}\,y^5\,\mathrm{D}_{2,\,2,\,2,\,2,\,2}(g)(0,\,0)$$

▶ *2.2.22. Megjegyzés.*

▼ *2.2.23. Implicit függvény tétel.*

```
> x:='x';y:='y';z:='z'; implicitdiff(x^2+y^2=1,y(x),x);
  implicitdiff(x^2+y^2=z^2,z(x,y),x);
  implicitdiff({x^2+y^2+z^2=1,x-y=z},{y(x),z(x)},y,x);
  solve({x^2+y^2+z^2=1,x-y=z},{y,z});
```

$$x := x$$

$$y := y$$

$$z := z$$

$$-\frac{x}{y}$$

$$\frac{x}{z}$$

$$-\frac{x+z}{y-z}$$

$$\{z = \mathrm{RootOf}(2\,x^2 - 2\,x\,_Z + 2\,_Z^2 - 1),\quad y = x - \mathrm{RootOf}(2\,x^2 - 2\,x\,_Z + 2\,_Z^2 - 1)\} \tag{2.2.23.1}$$

▶ *2.2.24. Inverz függvény tétel.*

▶ **2.2.25. Megjegyzés.*

▼ *2.2.26. Szélsőérték szükséges feltétele.*

```
> f:=cos(x+y)*cos(y+z)*cos(x+z); g:=Gradient(f,[x,y,z]);
```

$$f := \cos(x+y)\cos(y+z)\cos(x+z)$$

$$g := (-\sin(x+y)\cos(y+z)\cos(x+z) - \cos(x+y)\cos(y+z)\sin(x+z))\bar{e}_x + (-\sin(x+y)\cos(y+z)\cos(x+z) - \cos(x+y)\sin(y+z)\cos(x+z))\bar{e}_y + (-\cos(x+y)\sin(y+z)\cos(x+z) - \cos(x+y)\cos(y+z)\sin(x+z))\bar{e}_z \quad (2.2.26.1)$$

```
> subs([x=0,y=0,z=0],g); subs([x=Pi/2,y=Pi/2,z=Pi/2],g);
  subs([x=Pi/2,y=0,z=Pi/2],g);
```

$$0\bar{e}_x + 0\bar{e}_y + 0\bar{e}_z$$

$$0\bar{e}_x + 0\bar{e}_y + 0\bar{e}_z$$

$$0\bar{e}_x + 0\bar{e}_y + 0\bar{e}_z \quad (2.2.26.2)$$

▼ *2.2.27. Lokális szélsőérték elégséges feltétele.*

```
> with(Student[MultivariateCalculus]);
```

[*Add2DCoordinateNames*, *Add3DCoordinateNames*, *ApproximateInt*, *ApproximateIntTutor*, *CenterOfMass*, *ChangeOfVariables*, *CrossSection*, *CrossSectionTutor*, *DirectionalDerivative*, *DirectionalDerivativeTutor*, *FunctionAverage*, *Gradient*, *GradientTutor*, *Jacobian*, *LagrangeMultipliers*, *MultiInt*, *Revert*, *SecondDerivativeTest*, *SurfaceArea*, *TaylorApproximation*, *TaylorApproximationTutor*] (2.2.27.1)

```
> SecondDerivativeTest(f,[x,y,z]=[0,0,0]);
```

$$LocalMin = [\,],\ LocalMax = ([[0, 0, 0]]),\ Saddle = [\,] \quad (2.2.27.2)$$

```
> SecondDerivativeTest(f,[x,y,z]=[0,0,0],output=hessian);
```

(2.2.27.3)

$$\begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \tag{2.2.27.3}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,Pi/2,Pi/2]);
```

$$LocalMin = \left(\left[\left[\frac{1}{2}\,\pi,\ \frac{1}{2}\,\pi,\ \frac{1}{2}\,\pi \right] \right] \right),\ LocalMax = [\,],\ Saddle = [\,] \tag{2.2.27.4}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,Pi/2,Pi/2],output=
  hessian);
```

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{2.2.27.5}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,0,Pi/2]);
```

$$LocalMin = [\,],\ LocalMax = [\,],\ Saddle = \left(\left[\left[\frac{1}{2}\,\pi,\ 0,\ \frac{1}{2}\,\pi \right] \right] \right) \tag{2.2.27.6}$$

```
> SecondDerivativeTest(f,[x,y,z]=[Pi/2,0,Pi/2],output=
  hessian);
```

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix} \tag{2.2.27.7}$$

2.2.28. Lagrange-elv: feltételes szélsőérték keresése.

```
> LagrangeMultipliers(x*y,[x^2/8+y^2/2-1],[x,y]);
```

$$[2, 1],\ [-2, -1],\ [-2, 1],\ [2, -1] \tag{2.2.28.1}$$

->2.2.29. Feladat.

->2.2.30. Feladat.

->2.2.31. Feladat.

->2.2.32. Feladat.

->2.2.33. Feladat.

2.2.34. Felületek.

2.2.35. Felületek előállítási módjai.

2.2.36. Felületi normális, érintősík, felületi görbék.

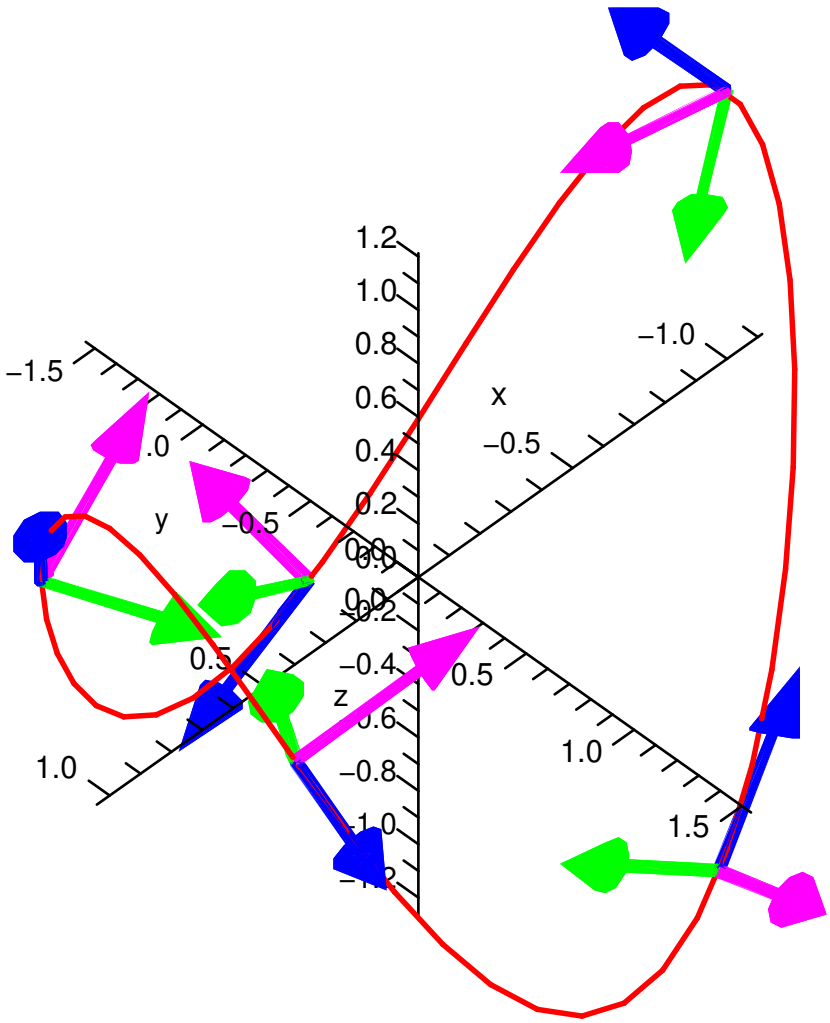
```
> TangentPlane(<s,t,s^2+t^2>,s=2,t=3);
```

$$(2+s)e_x+(3+t)e_y+(13+4\,s+6\,t)e_z \tag{2.2.36.1}$$

```
> TangentVector(<t,t^2,t^3>,t);
```

$$e_x+2\,te_y+3\,t^2e_z \tag{2.2.36.2}$$

```
> Student[VectorCalculus][SpaceCurveTutor]();
```



2.2.37. Approximáció.

```
> a:='a'; b:='b'; c:='c'; with(CurveFitting);
```

$$a := a$$

$$b := b$$

$$c := c$$

(2.2.37.1)

```
[BSpline, BSplineCurve, Interactive, LeastSquares,
    PolynomialInterpolation, RationalInterpolation, Spline,
    ThieleInterpolation]                                          (2.2.37.1)
> data:=[[0,13.3],[20,31.6],[50,85.5],[80,169.0],[100,246.0]]
  ;

 data := [[0, 13.3], [20, 31.6], [50, 85.5], [80, 169.0], [100, 246.0]]  (2.2.37.2)
> PolynomialInterpolation(data,x);
7.597222250 10^-7 x^4 - 0.0001288194449 x^3 + 0.4866944442 x      (2.2.37.3)
    + 0.02368777780 x^2 + 13.3
> PolynomialInterpolation(data,x,form='Lagrange');
0.000001662500000 (x - 20) (x - 50) (x - 80) (x - 100)            (2.2.37.4)
    - 0.00001097222222 x (x - 50) (x - 80) (x - 100)
    + 0.00003800000000 x (x - 20) (x - 80) (x - 100)
    - 0.00005868055556 x (x - 20) (x - 50) (x - 100)
    + 0.00003075000000 x (x - 20) (x - 50) (x - 80)
> LeastSquares(data,x,curve=a*x^2+b*x+c);
13.58761229 + 0.530547425866898403 x                              (2.2.37.5)
    + 0.01786658456486043 06 x^2
```

▶ *2.2.38. Newton-módszer.

▶ *2.2.39. Kapcsolat minimumfeladatokkal.

▶ *2.2.40. Az iránymenti csökkentés módszere.

▶ *2.2.41. Algebrai egyenletek megoldása.

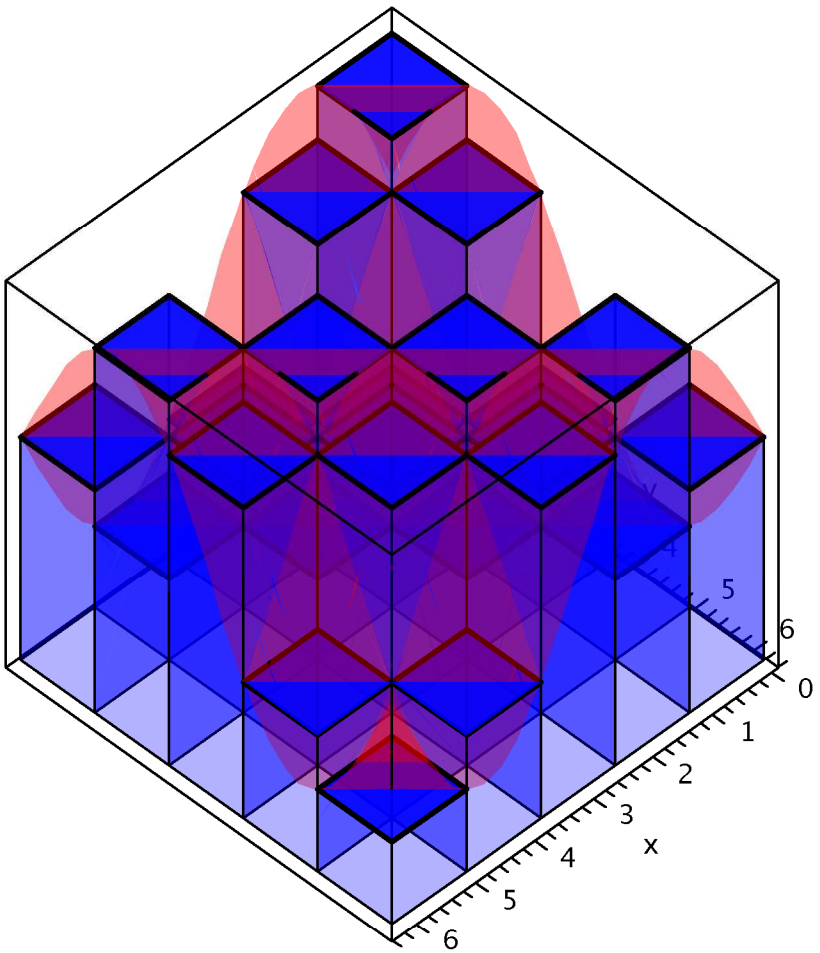
▼ 2.3. Integrálszámítás

```
> restart;with(VectorCalculus);
[&x, *, +, -, ., <,>, <|>, AddCoordinates, ArcLength, BasisFormat,   (2.3.1)
    Binormal, CrossProd, CrossProduct, Curl, Curvature, D, Del,
    DirectionalDiff, Divergence, DotProd, DotProduct, Flux,
    GetCoordinateParameters, GetCoordinates, Gradient, Hessian,
    Jacobian, Laplacian, LineInt, MapToBasis, Nabla, Norm, Normalize,
    PathInt, PrincipalNormal, RadiusOfCurvature, ScalarPotential,
    SetCoordinateParameters, SetCoordinates, SurfaceInt, TNBFrame,
```

Tangent, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, Wronskian, diff, evalVF, int, limit, series]

2.3.1. Többváltozós függvények integrálja.

```
> Student[MultivariateCalculus][ApproximateIntTutor]();
```



2.3.2. Tétel.
2.3.3. Következmény.
2.3.4. Tétel: az integrál egyértelműsége.
2.3.5. Tétel: az integrál linearitása.
2.3.6. Tétel: integrál és koordinátafüggvények.
2.3.7. Tétel: az integrál nemnegativitása.
2.3.8. Következmény: az integrál monotonitása.
**2.3.9. Tétel: Cauchy–kritérium.*

▶ *2.3.10. Segédtétel.

▶ 2.3.11. Tétel: az integrál mint halmazfüggvény additivitása.

▶ 2.3.12. Nullahalmazok.

▶ 2.3.13. Segédtétel.

▶ 2.3.14. Lebesgue-feltétel.

▶ 2.3.15. Következmény.

▶ 2.3.16. Tétel.

▶ 2.3.17. Megjegyzés.

▼ 2.3.18. Ismételt integrálás.

```
> int(x^2+y^2,[x,y]=Rectangle(-1..1,-2..2));
```

$$\frac{40}{3} \tag{2.3.18.1}$$

```
> int(x^2+y^2+z^4,[x,y,z]=Region(-1..1,-2..2,0..3));
```

$$\frac{2144}{5} \tag{2.3.18.2}$$

▶ 2.3.19. Abszolút integrálható függvények.

▶ 2.3.20. Tétel.

▼ 2.3.21. Definíció.

```
> int(x^2+y^2,[x,y]=Ellipse(2*x^2+3*y^2-1));
```

$$\frac{5}{144}\sqrt{6}\,\pi \tag{2.3.21.1}$$

▼ 2.3.22. Impropius integrál.

```
> int(exp(-x^2-y^2),[x,y]=Circle(<0,0>,R));
```

$$\pi - e^{-R^2}\pi \tag{2.3.22.1}$$

```
> limit(%,R=infinity);
```

$$\pi \tag{2.3.22.2}$$

▶ 2.3.23. Mérték.

▶ 2.3.24. Tétel.

▼ 2.3.25. Normál integrációs tartomány.

```
> int(x^2+y^2-z^2,[x,y,z]=Region(-R..R,-sqrt(R^2-x^2)..sqrt
  (R^2-x^2),
```

```
-sqrt(R^2-x^2-y^2)..sqrt(R^2-x^2-y^2)));
```

$$\frac{4}{15} R^5 \pi \tag{2.3.25.1}$$

▶ *2.3.26. Tétel.*

▶ *2.3.27. Példa: forgástest térfogata.*

▼ *2.3.28. Példa: tehetetlenségi nyomaték.*

```
> int(x^2+y^2,[x,y,z]=Region(-R..R,-sqrt(R^2-x^2)..sqrt(R^2-
  x^2),
  -sqrt(R^2-x^2-y^2)..sqrt(R^2-x^2-y^2)));
```

$$\frac{8}{15} R^5 \pi \tag{2.3.28.1}$$

▶ *2.3.29. Rademacher tétele.*

▶ *2.3.30. Integráltranszformációs formula.*

▼ *2.3.31. Példák: polár-, henger- és gömbi koordináták.*

```
> Jacobian(<r*cos(phi),r*sin(phi)>,[r,phi],'determinant'=
  true);
```

$$\begin{bmatrix} \cos(\phi) & -r\sin(\phi) \\ \sin(\phi) & r\cos(\phi) \end{bmatrix}, \cos(\phi)^2 r + r\sin(\phi)^2 \tag{2.3.31.1}$$

```
> simplify(%[2],trig);
```

$$r \tag{2.3.31.2}$$

```
> Jacobian(<r*cos(phi),r*sin(phi),z>,[r,phi,z],'determinant'=
  true);
```

$$\begin{bmatrix} \cos(\phi) & -r\sin(\phi) & 0 \\ \sin(\phi) & r\cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cos(\phi)^2 r + r\sin(\phi)^2 \tag{2.3.31.3}$$

```
> simplify(%[2],trig);
```

$$r \tag{2.3.31.4}$$

```
> Jacobian(<r*cos(phi)*cos(theta),r*sin(phi)*cos(theta),r*sin
  (theta)>,
  [r,phi,theta],'determinant'=true);
```

$$\begin{bmatrix} \cos(\phi)\cos(\theta) & -r\sin(\phi)\cos(\theta) & -r\cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & r\cos(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) \\ \sin(\theta) & 0 & r\cos(\theta) \end{bmatrix}, \tag{2.3.31.5}$$

$$\cos(\phi)^2\cos(\theta)^3 r^2 + \sin(\phi)^2\cos(\theta)^3 r^2$$
$$+ r^2\sin(\phi)^2\cos(\theta)\sin(\theta)^2 + r^2\cos(\phi)^2\cos(\theta)\sin(\theta)^2$$

```
> simplify(%[2],trig);
```

$$\cos(\theta) r^2 \tag{2.3.31.6}$$

```
> with(Student[MultivariateCalculus]);
```

$$[Add2DCoordinateNames, Add3DCoordinateNames, \tag{2.3.31.7}$$
$$ApproximateInt, ApproximateIntTutor, CenterOfMass,$$
$$ChangeOfVariables, CrossSection, CrossSectionTutor,$$
$$DirectionalDerivative, DirectionalDerivativeTutor,$$
$$FunctionAverage, Gradient, GradientTutor, Jacobian,$$
$$LagrangeMultipliers, MultiInt, Revert, SecondDerivativeTest,$$
$$SurfaceArea, TaylorApproximation, TaylorApproximationTutor]$$

```
> ChangeOfVariables(x^2+y^2,['cartesian'[x,y],'polar'[r,phi]]
  );
```

$$r^2\cos(\phi)^2 + r^2\sin(\phi)^2 \tag{2.3.31.8}$$

```
> simplify(%,trig);
```

$$r^2 \tag{2.3.31.9}$$

```
> ChangeOfVariables(Int(Int(x^2+y^2,x),y),
  ['cartesian'[x,y],'polar'[r,phi]]);
```

$$\iint r^3 \,d\phi\, dr \tag{2.3.31.10}$$

```
> ChangeOfVariables(x^2+y^2-z^2,['cartesian'[x,y,z],
  'cylindrical'[r,phi,z]]);
```

$$r^2\cos(\phi)^2 + r^2\sin(\phi)^2 - z^2 \tag{2.3.31.11}$$

```
> simplify(%,trig);
```

$$-z^2 + r^2 \tag{2.3.31.12}$$

```
> ChangeOfVariables(Int(Int(Int(x^2+y^2-z^2,x),y),z),
  ['cartesian'[x,y,z],'cylindrical'[r,phi,z]]);
```

$$\iiint \left(-z^2 + r^2\right) r \,d\phi\, dr\, dz \tag{2.3.31.13}$$

▶ ***2.3.32. Tétel: paraméteres integrálok differenciálása.***

▶ ***2.3.33. Pályák.***

▶ ***2.3.34. Tétel.***

▼ ***2.3.35. Tétel.***

```
> v:=<cos(t),sin(t),t>; diff(v,t); %.%; simplify(%,trig);
```

$$v := (\cos(t))e_x + (\sin(t))e_y + (t)e_z$$

$$-\sin(t)e_x + (\cos(t))e_y + e_z$$

$$1 + \sin(t)^2 + \cos(t)^2$$

$$2 \qquad (2.3.35.1)$$

▶ 2.3.36. Példák.

▼ 2.3.37. Sima pályák természetes paraméterezése.

```
> s=int(sqrt(%),t=0..T); solve(%,T); subs(t=%,v);
  diff(%,s); %.%; simplify(%,trig);
```

$$s = \sqrt{2}\ T$$

$$\frac{1}{2}\ s\sqrt{2}$$

$$\left(\cos\left(\frac{1}{2}\ s\sqrt{2}\right)\right)e_x + \left(\sin\left(\frac{1}{2}\ s\sqrt{2}\right)\right)e_y + \frac{1}{2}\ s\sqrt{2}\,e_z$$

$$-\frac{1}{2}\sin\left(\frac{1}{2}\ s\sqrt{2}\right)\sqrt{2}\,e_x + \frac{1}{2}\cos\left(\frac{1}{2}\ s\sqrt{2}\right)\sqrt{2}\,e_y + \frac{1}{2}\sqrt{2}\,e_z$$

$$\frac{1}{2} + \frac{1}{2}\sin\left(\frac{1}{2}\ s\sqrt{2}\right)^2 + \frac{1}{2}\cos\left(\frac{1}{2}\ s\sqrt{2}\right)^2$$

$$1 \qquad (2.3.37.1)$$

▼ 2.3.38. Integrálás felületeken és felszín.

```
> PathInt(x^2+y^2+z^2,[x,y,z]=Path(<t,t^2,t^3>,t=0..2));
  evalf(%);
```

$$\frac{2}{36971235}\ \frac{1}{\sqrt{-2+\mathrm{I}\sqrt{5}}\left(2+\mathrm{I}\sqrt{5}\right)}\left(\sqrt{161}\left(\right.\right.$$

$$768697076\sqrt{-2+\mathrm{I}\sqrt{5}}$$

$$+384348538\,\mathrm{I}\sqrt{-2+\mathrm{I}\sqrt{5}}\ \sqrt{5}$$

$$-20010\sqrt{9-4\,\mathrm{I}\sqrt{5}}\ \sqrt{9+4\,\mathrm{I}\sqrt{5}}\ \mathrm{EllipticF}\left(2\sqrt{-2+\mathrm{I}\sqrt{5}}\,,\right.$$

$$\left.\left.\frac{1}{3}\sqrt{-1+4\,\mathrm{I}\sqrt{5}}\right)\right.$$

$$-3523\,\mathrm{I}\sqrt{9-4\,\mathrm{I}\sqrt{5}}\,\sqrt{9+4\,\mathrm{I}\sqrt{5}}\,\mathrm{EllipticF}\left(2\sqrt{-2+\mathrm{I}\sqrt{5}},\ \frac{1}{3}\sqrt{-1+4\,\mathrm{I}\sqrt{5}}\right)\sqrt{5}$$

$$+12964\sqrt{9-4\,\mathrm{I}\sqrt{5}}\,\sqrt{9+4\,\mathrm{I}\sqrt{5}}\,\mathrm{EllipticE}\left(2\sqrt{-2+\mathrm{I}\sqrt{5}},\ \frac{1}{3}\sqrt{-1+4\,\mathrm{I}\sqrt{5}}\right)\Bigg)\Bigg)$$

$$263.8435852-7.095521418\,10^{-8}\,\mathrm{I} \qquad (2.3.38.1)$$

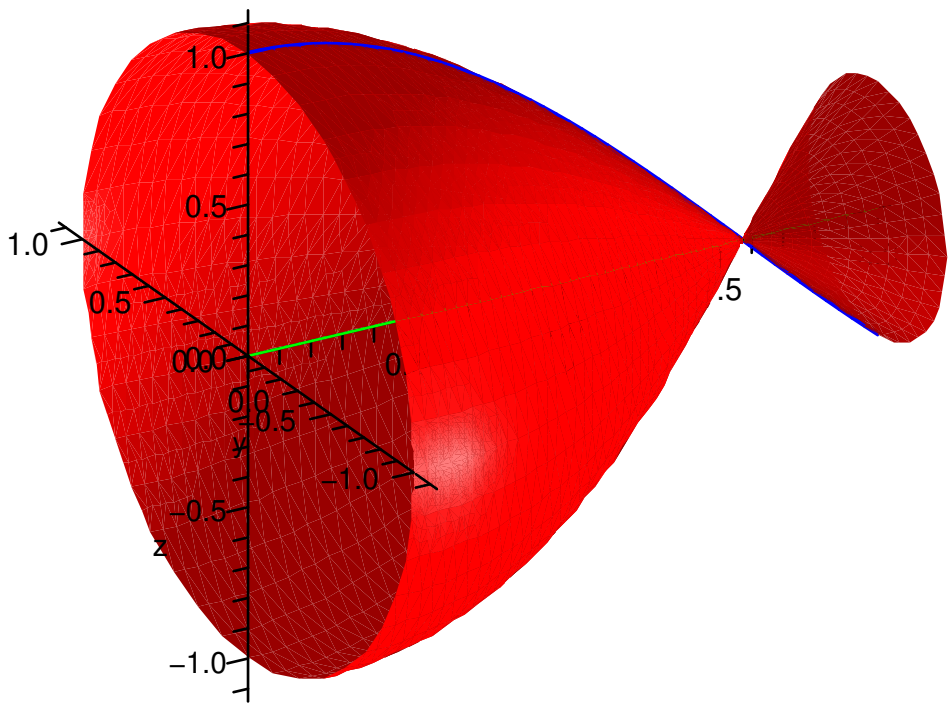
```
> SurfaceInt(x+y+z,[x,y,z]=Surface(<s,t,4-s*t>,
  [s,t]=Triangle(<0,0>,<0,1>,<1,0>))); evalf(%);
```

$$\int_0^1 \Bigg(\frac{1}{3}\left(1+s^2\right)^{3/2}s-\frac{1}{4}\,s\ln\left(1+s^2\right)-\frac{1}{3}\left(1+s^2\right)^{3/2}-\frac{1}{4}\ln\left(1+s^2\right)s^3-\ln\left(1+s^2\right)s^2-\ln\left(1+s^2\right)-\frac{1}{3}\left(2+2\,s^2-2\,s\right)^{3/2}s+2\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)s^2+\frac{1}{3}\left(2+2\,s^2-2\,s\right)^{3/2}-\frac{1}{2}\,s^2\sqrt{2+2\,s^2-2\,s}-\frac{3}{2}\,s\sqrt{2+2\,s^2-2\,s}+\frac{1}{2}\,s\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)+\frac{1}{2}\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)s^3+2\sqrt{2+2\,s^2-2\,s}+2\ln\left(-s+1+\sqrt{2+2\,s^2-2\,s}\right)\Bigg)\mathrm{d}s$$

$$2.646495329 \qquad (2.3.38.2)$$

2.3.39. Példa: forgásfelület felszíne.

```
> Student[Calculus1][SurfaceOfRevolutionTutor]();
```



▶ 3. Függvénysorozatok és függvénysorok