MEASURABILITY IMPLIES CONTINUITY FOR SOLUTIONS OF FUNCTIONAL EQUATIONS — EVEN WITH FEW VARIABLES

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Abstract. It is proved that — under certain conditions — measurable solutions f of the functional equation

$$f(x) = h(x, y, f(g_1(x, y)), \dots, f(g_n(x, y))), \quad (x, y) \in D \subset \mathbb{R}^s \times \mathbb{R}^l$$

are continuous, even if $1 \le l \le s$. As a tool we introduce new function classes which — roughly speaking — interpolate between continuous and Lebesgue measurable functions. Connection between these classes are also investigated.

1. Introduction

In connection with his fifth problem Hilbert [5] suggested that although the method of reduction to differential equations makes it possible to solve functional equations in an elegant way, the inherent differentiability assumptions are typically unnatural (see [2]). Such shortcomings can be overcome by appealing to regularity theorems.

In this spirit the following general regularity problem of non-composite functional equations with several variables was formulated by the author and included by Aczél among the most important open problems on functional equations (see Aczél [1] and Járai [7]):

1.1. Problem. *Let X and Z be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $X \times X$. Let $f : X \to Z$, $g_i : D \to X$ $(i = 1, 2, \dots, n)$ and $h : D \times Z^{n+1} \to Z$ be functions. Suppose that*

(1)

$$f(x) = h(x, y, f(y), f(g_1(x, y)), \dots, f(g_n(x, y))) \text{ whenever } (x, y) \in D;$$

(2) h is analytic;

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(3) *g_i is analytic and for each $x \in X$ there exists a y for which $(x,y) \in D$ and $\frac{\partial g_i}{\partial y}(x,y)$ has rank s $(i = 1, 2, \dots, n)$.*

Is it true that every f which is measurable or has the Baire property is analytic?

The following steps can be used:

(I) Measurability implies continuity.

(II) Baire property implies continuity.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p times continuously differentiable solutions are $p+1$ times continuously differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

We note that in order to obtain $f \in \mathcal{C}^p$ it is usually enough to suppose only that the given functions h and g_i are in $\mathcal{C}^p$ (if $2 \le p \le \infty$) or in $\mathcal{C}^{p+1}$ (if $p = 0$ or $p = 1$). The complete answer to the problem above is not known. The author discussed this problem in several papers and solved problems corresponding to (I), (II), (IV) and (V) (see [7]), and under some additional compactness condition (III) (see [8]). References can be found in the survey paper [14]. There are some partial results in connection with (VI). Moreover, other properties of solutions such as having locally bounded variation or local Hölder continuity are also discussed (see [12] and references in [14]). It is possible to extend these results to manifolds, and the $\mathcal{C}^\infty$-part of the problem is completely solved on compact manifolds [11]. The most applicable results are treated in the booklet [10].

Regularity theorems of the type "locally integrable solutions are infinitely many times differentiable" can be obtained using distributions. The essence of the method is to prove that solutions in the distribution sense satisfy a differential equation having only infinitely many times differentiable solutions. This idea was used by Światak [18] to prove general regularity results for the functional equation

$$\sum_{i=1}^{n} h_i(x,y) f(g_i(x,y)) = h(x, f(g_{n+1}(x)), \dots, f(g_m(x))) + h_0(x,y),$$

where f is the only unknown function. Roughly speaking, she applies a partial differential operator in y to the equation in the distribution sense. Of course, the nonlinear term on the right hand side disappears. If, after substituting a fixed y_0, we are fortunate enough to obtain a hypoelliptic partial differential equation, then by the regularity theory of partial differential equations all distribution solutions are in $\mathcal{C}^\infty$. For the exact details of how to overcome the difficulties and for applications see her paper [18].

Further references about regularity theorems for functional equations can be found in the survey paper [14]. Some other papers concerning the distribution method are also referred to there.

The above equation of Światak is "almost linear", so, formally, it is much less general than equation (1). However her theorems can be applied even if the rank of $\frac{\partial g_i}{\partial y}$ is much smaller than the dimension of the domain of the unknown function f. Roughly speaking, the present author's results, quoted above, may be applied to prove regularity of a solution f having s variables, only if there are at least $2s$ variables in the functional equation. The method of Światak may be applied even if there are only $s+1$ variables. This is the minimal number of variables: in Hilbert's paper [5] there is an example that for "one variable" functional equations (this may mean an s-dimensional vector variable) no regularity theorem holds. So the results of Światak suggest that the rank condition in the problem above is too strong, and the results concerning the above problem can be extended for a much more general case. Generalizing our method we may hope to obtain regularity results for general *nonlinear* functional equations; which seems to be impossible using the method of Światak based on Schwartz distributions. We may not hope to be so lucky that with one substitution $y = y_0$ we have $g_i(x, y_0) \equiv x$ for all i; a very strong condition. The somewhat artificial condition of hypoellipticity also has to disappear. What seems to be most important is to prove "measurability implies continuity" type results, because by the method of Światak we may only start with locally integrable solutions — a consequence of the distribution method. To the best knowledge of the author such "measurability implies continuity" type results without the strong rank condition in (3) or some abstract version of it are known only for very special equations such as for example the equation

$$f(x) = \sum_{i=1}^{m} \mu_i f(x + y e_i), \quad x \in \mathbb{R}^n,\, y \in \mathbb{R}$$

($\mu_i \in \mathbb{R}$, $e_i \in \mathbb{R}^n$ are fixed) in the paper of McKiernan [15]. The proof there is based on algebraic properties of the solutions.

In this paper we will prove a "measurability implies continuity" type result for the general explicit nonlinear functional equation (1) without the strong rank condition in (3) on the inner functions. In the spirit of the "bootstrap" method corresponding to steps (I)–(VI) we introduce a sequence of properties, which — roughly speaking — interpolate between measurability and continuity. This sequence of properties gives a stairway to climb up from measurability to continuity. First we will investigate the basic properties of the new notions. Then the regularity theorem will be proved. An example is given how to apply the theorem in nontrivial cases. A refinement of the theorem is also proved. Finally, further properties of the new notions are investigated.

2. The new notions

2.1. Notation. If f is a function then $\operatorname{rng} f$ denotes the *range* of f. All *normed spaces* are supposed to be real; the norm is denoted by $|\ |$. Only operator norms will be denoted by $\|\ \|$. If $f : D \to Y$ is a function mapping an open subset of a normed space into a normed space, then f' will denote the *derivative* of f. If $D \subset X_1 \times X_2 \times \ldots \times X_n$, we will use the *partial sets*

$$D_{x_i} = \big\{(x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_n) : (x_1, ..., x_n) \in D\big\}.$$

The *partial functions* $f_{x_i} : D_{x_i} \to Y$ are defined by

$$f_{x_i}(x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_n) = f(x_1, \ldots , x_n)$$

whenever $(x_1, \ldots , x_n) \in D$. (Notice that x_i is held constant in f.) Also $D_{x_{i_1},\ldots,x_{i_r}}$ and $f_{x_{i_1},\ldots,x_{i_r}}$ are defined similarly. Now, if X_i and Y are normed spaces and

$$D_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}$$

is an open subset of X_i we define the *partial derivative* denoted by

$$\partial_i f, \quad \partial_{x_i} f \quad \text{or} \quad \frac{\partial f}{\partial x_i}$$

as the derivative of $f_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}$. Other notions of calculus are used in the usual way.

If x, y are points of a metric space and $\alpha > 0$, we say that x and y are α-near if their distance is less than α. Similarly, if x and y are points of a uniform space and α is a relation from the uniformity we say that x and y are α-near if $(x, y) \in \alpha$. In a metric space the closed ball having radius $r \geq 0$ and center x will be denoted by $\mathbb{B}_r(x)$.

Concerning measure theory, we follow the terminology of Federer [4]. Hence a *measure* means a countably subadditive extended real valued nonnegative function defined on all subsets of a set; this is called outer measure in other terminology. By a *Radon measure* we mean a locally finite measure μ defined on a Hausdorff space X, with the following properties:

(1) Every open subset V of X is measurable and

$$\mu(V) = \sup\{\mu(K) : K \subset V,\ K \text{ compact}\};$$

(2) If A is any subset of X, then

$$\mu(A) = \inf\{\mu(V) : A \subset V,\ V \text{ open}\}.$$

λ^n will denote the Lebesgue measure on $\mathbb{R}^n$, and χ^m will denote the m-dimensional Hausdorff measure on a metric space.

Assuming that μ is a measure on X, $A \subset X$ and Y is a topological space, we say that the function f is *measurable over* A if f is defined at almost every point of A, the range of f is contained in Y and $A \cap f^{-1}(W)$ is measurable whenever W is an open subset of Y. If μ is a Radon measure on X and f maps almost all of X into a topological space Y then we say that f is a *Lusin function*, if for each measurable subset A of X having finite measure and for each $\varepsilon > 0$ there is a compact subset C of A such that $\mu(A \backslash C) < \varepsilon$ and $f|C$ is continuous. In this setting Lusin's theorem says that if Y is a second countable topological space and μ is a Radon measure then every function which is measurable over X is a Lusin function. The proof can be found in [17], 8.2 by Oxtoby.

We refer the reader to Federer [4] concerning the proof of other measure theoretical results used here.

2.2. Definition. Let X be a set, Y a metric space, and $f : X \to Y$ be a function. Let U be a Hausdorff space with the Radon (outer) measure μ, and P a topological space, the "parameter space" with a given point $p_0 \in P$. Let φ be a function from $U \times P$ into X. We will think of φ as a surface $\varphi_p : u \mapsto \varphi(u,p)$ for each p, depending on the parameter p.

Lusin's theorem and generalizations of Steinhaus' theorem [9] suggest that the following condition is connected with measurability:

(L) *For each $\varepsilon > 0$, each $\sigma > 0$ and for each compact subset $C \subset U$ there exists a neighborhood P_0 of p_0 such that if $p \in P_0$ then*

$$\mu\{u \in C : \operatorname{dist}(f(\varphi(u,p)), f(\varphi(u,p_0))) \geq \sigma\} \leq \varepsilon.$$

The condition above can be reformulated in the following sequential way:

(S) *For each $\sigma > 0$, for each compact subset $C \subset U$ and for each sequence $p_m \to p_0$*

$$\mu\{u \in C : \operatorname{dist}(f(\varphi(u,p_m)), f(\varphi(u,p_0))) \geq \sigma\} \to 0.$$

In this form the condition strongly resembles convergence in measure. Riesz theorem suggests the following condition:

(R) *For each sequence $p_m \to p_0$ there exists a subsequence p_{m_i} such that for almost all $u \in U$ we have*

$$f(\varphi(u,p_{m_i})) \to f(\varphi(u,p_0)).$$

This condition resembles the following condition treated by Trautner in a special case for characteristic functions of measurable sets (see remark below):

(T) *For each sequence* $p_m \to p_0$ *and for almost all* $u \in U$ *there exists a subsequence* p_{m_i} *such that*

$$f(\varphi(u, p_{m_i})) \to f(\varphi(u, p_0)).$$

To investigate the connection between these conditions we need some kind of measurability condition:

(M) $u \mapsto f(\varphi(u, p_0))$ *is* μ *measurable.*

It is clear that conditions (L) and (S) have meaning also if the values of f are in a uniform space Y; simply σ has to be replaced by a reflexive symmetric relation from the uniformity of Y and we have to consider the set of those points u for which the two values of f are not σ-near. Condition (R) has the advantage that it has meaning even if Y is only a topological space. The same is true for (T) and (M). It seems that (T) has no advantage over (R).

We will often check condition (L) [(S), (R), (T), (M)] locally. If for each $u_0 \in U$ there is a neighborhood U_0 of u_0 and P_0 of p_0 such that $\varphi|U_0 \times P_0$ satisfies (L) [(S)], then φ also satisfies (L) [(S)]. To see this, we will choose a finite covering of C by open sets having finite measure and we will apply (L) [(S)] to a sufficiently good inner approximation of these open sets by compact sets: Let us choose for each $x \in C$ a neighborhood U_x of x and a neighborhood P_x of p_0 such that $\varphi|U_x \times P_x$ satisfies (L). Shrinking U_x if necessary we may suppose that U_x is open and has finite μ measure. Let $U_{x_1}, \dots, U_{x_r}$ be a finite subcovering of C, let $\varepsilon, \sigma > 0$ and let us choose compact sets $C_i \subset U_{x_i}$ for which $\mu(U_{x_i} \setminus C_i) < \varepsilon/(2r)$. Choosing a neighborhood P_0 of p_0 for which $P_0 \subset \cap_{i=1}^r P_{x_i}$ such that the sets

$$R_i(p) = \{u \in C_i : \operatorname{dist}(f(\varphi(u, p)), f(\varphi(u, p_0))) \geq \sigma\}$$

have μ measure less than $\varepsilon/(2r)$ for each $p \in P_0$, we obtain that

$$\mu\{u \in C_i : \operatorname{dist}(f(\varphi(u, p)), f(\varphi(u, p_0))) \geq \sigma\} \leq \varepsilon,$$

because this set is covered by $\cup_{i=1}^r R_i(p) \cup \cup_{i=1}^r (U_{x_i} \setminus C_i)$. Similarly, if $p_m \to p$ and (S) is satisfied by $\varphi|U_x \times P_x$, then for given $\varepsilon, \sigma > 0$ for $i = 1, \dots, r$ we obtain an M_i such that for $m \geq M_i$ we have $p_m \in P_{x_i}$ and $R_i(p_m)$ has μ measure less than $\varepsilon/(2r)$ for each $m \geq M_i$. Hence for $m \geq M = \max_{1 \leq i \leq r} M_i$ we have

$$\mu\{u \in C_i : \operatorname{dist}(f(\varphi(u, p)), f(\varphi(u, p_0))) \geq \sigma\} \leq \varepsilon.$$

Similarly, if for each $u_0 \in U$ there is a neighborhood U_0 of u_0 and P_0 of p_0 such that $\varphi|U_0 \times P_0$ satisfies (R) [(T), (M)], then supposing that U is a Lindelöf space we have that φ satisfies (R) [(T), (M)]. For (R) this follows using the diagonal process. Countably many of the sets U_0 cover U. Let us enumerate these open sets, and let us consider repeatedly sub-sub-...-sequences of the sequence p_m. The diagonal process gives a subsequence, for which the convergence is satisfied almost everywhere. In the case of (T) and (M) the statement is trivial.

Let X be an open subset of $\mathbb{R}^n$ and $0 \leq k \leq n$. The class of all functions f for which the condition (L) [(S), (R), (T), (M)] is satisfied whenever U is an open subset of $\mathbb{R}^k$, $\mu = \lambda^k$, P is an open subset of some Euclidean space, $p_0 \in P$ and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$-function for which φ_p is an immersion of U into X for each $p \in P$, will be denoted by $\mathcal{L}_k(X,Y)$ or shortly by $\mathcal{L}_k$ [$\mathcal{S}_k$, $\mathcal{R}_k$, $\mathcal{T}_k$, $\mathcal{M}_k$]. (Recall, that a $\mathcal{C}^1$ mapping of U into X is an immersion if and only if its derivative is an injective linear mapping for each point of U. For $k = 0$, take $\mathbb{R}^0 = \{0\}$ and $\lambda^0(\{0\}) = 1$, i.e. λ^0 is the counting measure on $\mathbb{R}^0$. A function $\varphi : \{0\} \times P \to X$ is a $\mathcal{C}^1$ function if and only if $p \mapsto \varphi(0,p)$ is a $\mathcal{C}^1$ function. Any function mapping a subset of $\mathbb{R}^0$, i.e. $\emptyset$ or $\{0\}$ into X is considered an immersion.) In the first two cases we suppose that the values of f are in a uniform space, in the other three that they are in a topological space. It is clear that $f \in \mathcal{M}_k$ if and only if the condition

(M′) *$f \circ \psi$ is μ measurable*

is satisfied for $\mu = \lambda^k$ whenever ψ is an immersion of some open subset U of $\mathbb{R}^k$ into X.

2.3. Remarks. (1) For our purposes, the function class $\mathcal{R}_k(X,Y)$ will be the most convenient one, because we want to avoid supposing that Y is a uniform space. It is even more important, that using $\mathcal{R}_k(X,Y)$ we can avoid supposing uniform continuity for the given functions in our regularity theorems and it is enough to suppose continuity. The classes $\mathcal{M}_k$ and $\mathcal{L}_k$ will also play a role. Our main results will show that, roughly speaking, solutions f of a functional equation from $\mathcal{R}_{k+1}$ are also in $\mathcal{R}_k$. We will prove that $\mathcal{R}_0$ is the class of continuous functions, and that all measurable functions $f : X \to Y$ from the open subset $X \subset \mathbb{R}^n$ into some second countable space Y are in $\mathcal{R}_n$. Hence, step-by-step, measurability of solutions implies their continuity.

(2) In his paper [19] Trautner proved that for a Lebesgue measurable subset M of $[a,b] \subset \mathbb{R}$ with positive Lebesgue measure and for a sequence $p_m \in [a,b]$ there exists an $u \in \mathbb{R}$ and a subsequence p_{m_s} such that $p_{m_s} + u \in M$. This follows from the fact that a Lebesgue measurable function is in $\mathcal{T}_1$. Indeed, let us replace p_m with a subsequence converging to a point $p_0 \in [a,b]$. Let $f = \xi_M$ be the characteristic function of M, and let $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be $\varphi(u,p) = u - p$. From $\xi_M \in \mathcal{T}_1$ it follows that for almost all $u \in M + p_0$ there exists a subsequence p_{m_s} such that

$$\xi_M(u - p_{m_s}) \to \xi_M(u - p_0) = 1.$$

This means that $u + p_{m_s} \in M$ for large enough s.

Trautner used his theorem — among others — to give a new proof of the well-known result of Steinhaus, that measurable additive mappings of $\mathbb{R}$ into itself are continuous.

Trautner's method was generalized to locally compact groups and to an even more general setting by Grosse-Erdmann [3]. His results can be applied to prove that for the functional equation

$$f(g(x,y)) = h(y, f_1(x))$$

with unknown functions f, f_1 — under suitable conditions — measurability of f_1 implies the continuity of f. He applies his abstract results for the case where $(x,y) \in D$, some open subset of $\mathbb{R}^n \times \mathbb{R}^n$, $g : D \to \mathbb{R}^n$ and $\det \frac{\partial g}{\partial x}$ and $\det \frac{\partial g}{\partial y}$ are nonzero. His method has the advantage that one only needs the continuity of h with respect to the second variable. Note that substituting $t = g(x,y)$ we have locally

$$f(t) = h(y, f_1(g_1(t,y)));$$

compare this with Problem 1.1. Condition (T) does not seem to be strong enough for us to obtain "measurability implies continuity" type results for the more general equation in Problem 1.1.

(3) The class $\mathcal{L}_k$ [$\mathcal{S}_k$, $\mathcal{R}_k$, $\mathcal{T}_k$, $\mathcal{M}_k$] remains the same if we suppose only that (L) [(S), (R), (T), (M)] is satisfied whenever U is an open subset of $\mathbb{R}^k$, $\mu = \lambda^k$, P is an open subset of some Euclidean space, $p_0 \in P$ and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$-function for which φ_p is an embedding (i.e., an immersion which is a homeomorphism of its domain onto its range) of U into X for each $p \in P$. This easily follows from the locality principle mentioned in the definition.

Similarly, supposing only that φ_{p_0} is an immersion, the resulting class $\mathcal{L}_k$ [$\mathcal{S}_k$, $\mathcal{R}_k$, $\mathcal{T}_k$, $\mathcal{M}_k$] remains the same.

(4) In condition (L) [(S)] the words "for each compact subset C of U" can be equivalently replaced by "for each σ-finite measurable subset C of U". This easily follows using inner approximation by compact sets.

We start with the investigation of the simplest connections between the classes $\mathcal{L}_k$, $\mathcal{S}_k$, $\mathcal{R}_k$, $\mathcal{T}_k$ and $\mathcal{M}_k$.

2.4. Theorem. *With the notation of the definition above, condition (L) implies condition (S). If the point p_0 has a countable base of neighborhoods then (L) follows from (S). If the uniformity of Y has a countable base and μ is σ-finite, then (S) implies (R). (R) always implies (T). If Y is a uniform space with a countable base of topology, (R) is satisfied, and (M) is satisfied for all $p_0 \in P$, then (S) is satisfied, too. Hence, if Y is a separable metric space, then $\mathcal{L}_k = \mathcal{S}_k \subset \mathcal{R}_k \subset \mathcal{T}_k$ and $\mathcal{L}_k \cap \mathcal{M}_k = \mathcal{S}_k \cap \mathcal{M}_k = \mathcal{R}_k \cap \mathcal{M}_k$.*

Proof. It is easy to see that (L) implies (S) and if the point p_0 has a countable base of neighborhoods then (L) follows from (S). Condition (R) implies (T) trivially.

The proof that if Y is a metric space and μ is σ-finite then (S) implies (R), mimics the proof of the classical Riesz' theorem: Let C be an arbitrary compact subset of U and let us choose a sequence $\sigma_i \downarrow 0$. We may choose a subsequence p_{m_i} such that the set

$$\{u \in C : \operatorname{dist}(f(\varphi(u, p_{m_i})), f(\varphi(u, p_0))) \geq \sigma_i\}$$

has μ (outer) measure less than 2^{-i}. Let A_i denote a μ-hull of this set. Now if u is not in the zero set $\cap_{j=1}^{\infty} \cup_{i=j}^{\infty} A_i$, then

$$\operatorname{dist}(f(\varphi(u, p_{m_i})), f(\varphi(u, p_0))) < \sigma_i$$

for all $i \geq j$ for some j. Let us choose a countable almost cover of U by compact sets $C_1, C_2, \ldots$. Let us consider repeatedly sub-sub-...-sequences of the sequence p_m. The diagonal process gives a subsequence, for which the convergence is satisfied almost everywhere. The same proof works in the case of a uniform space having a countable base of uniformity.

Now suppose that Y is a separable metric space. If f satisfies (M) for every $p_0 \in P$, then we obtain that $u \mapsto \varphi(u, p)$ is μ measurable for all $p \in P$. Using that Y is separable, we obtain that for any pair $p, p' \in P$ the mapping $u \mapsto (f(\varphi(u, p)), f(\varphi(u, p')))$ of U into $Y \times Y$ is measurable too. This implies that for each pair $p, p' \in P$ the mapping

$$u \mapsto \operatorname{dist}(f(\varphi(u, p)), f(\varphi(u, p')))$$

is measurable. Now suppose that (S) is not satisfied by φ with $p_0 \in P$. This means that there is a sequence $p_m \to p_0$, $\sigma > 0$, $\varepsilon > 0$, and a compact set $C \subset U$ such that the measure of the *measurable* sets

$$\{u \in C : \operatorname{dist}(f(\varphi(u, p_m)), f(\varphi(u, p_0))) \geq \sigma\}$$

is greater than ε for infinitely many m. Let us choose a subsequence p_{m_i} for which each of the measurable sets

$$A_i = \{u \in C : \operatorname{dist}(f(\varphi(u, p_{m_i})), f(\varphi(u, p_0))) \geq \sigma\}$$

has measure $\geq \varepsilon$. Then for an arbitrary subsequence $p_{m_{i_j}}$ for any u from the measurable set $\cap_{k=1}^{\infty} \cup_{j=k}^{\infty} A_{i_j}$ having measure $\geq \varepsilon$ we have

$$f(\varphi(u, p_{m_{i_j}})) \not\to f(\varphi(u, p_0)).$$

This contradicts to that f satisfies (R). Hence we have

$$\mu\{u \in C : \operatorname{dist}(f(\varphi(u, p_m)), f(\varphi(u, p_0))) \geq \sigma\} \to 0.$$

The same proof works for second countable uniform spaces.

2.5. Theorem. *Let Y be a topological space and X an open subset of $\mathbb{R}^n$. Then $\mathcal{M}_0(X,Y) = Y^X$ and $\mathcal{R}_0(X,Y) = \mathcal{T}_0(X,Y) = \mathcal{C}(X,Y)$, the class of continuous functions from X into Y. If Y is a uniform space then also $\mathcal{L}_0(X,Y) = \mathcal{S}_0(X,Y) = \mathcal{C}(X,Y)$.*

Proof. We will use the notation of the definition. It is trivial that $\mathcal{M}_0$ contains all functions from X into Y.

Now let us prove that any continuous function $f : X \to Y$ is in $\mathcal{R}_0$, hence also in $\mathcal{T}_0$. There are only two cases, $U = \emptyset$ or $U = \{0\}$. In the first case, there is nothing to prove; in the second case we may choose $p_{m_k} = p_k$.

The converse is proved indirectly: if $f \in \mathcal{T}_0$, but not continuous, then there exists an $x_0 \in X$, a sequence $x_n \to x_0$, and a neighborhood W of $f(x_0)$ such that $f(x_n) \notin W$. Let $U = \{0\}$, $P = X$, $p_0 = x_0$, $\varphi(0,p) = p$ for $p \in P$. Choosing a subsequence of the sequence $p_m = x_m$ for which

$$f(\varphi(0, p_{m_k})) = f(x_{m_k}) \to f(x_0) = f(\varphi(0, p_0)),$$

we obtain a contradiction.

If Y is a uniform space, f is continuous, and $C = \{0\}$ then every $p_0 \in P$ has a neighborhood P_0 such that if $p \in P_0$, then $f(\varphi(0,p))$ and $f(\varphi(0,p_0))$ are close enough, whence $f \in \mathcal{L}_0 \subset \mathcal{S}_0$.

Supposing f is discontinuous at an $x_0 \in X$, and choosing $U = C = \{0\}$, $P = X$, $p_0 = x_0$, $\varphi(0,p) = p$ for $p \in P$, we obtain a sequence $p_m \to p_0$ such that $f(\varphi(0,p_m))$ and $f(\varphi(0,p_0))$ are not close, which shows that f is not in $\mathcal{S}_0$.

We will prove that Lebesgue measurable functions over an open subset X of $\mathbb{R}^n$ are in $\mathcal{R}_n$. To make the connection with earlier results in [7] clear, we do the main part of the proof in the following abstract setting:

2.6. Theorem. *Let P be a topological space, U and X Hausdorff spaces with finite Radon (outer) measures μ and ν, respectively. Suppose that $\varphi : U \times P \to X$ is a continuous function with the following property:*

(1) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $p \in P$, $B \subset U$, $\mu(B) \geq \varepsilon$ then $\nu(\varphi_p(B)) \geq \delta$.

Suppose, moreover, that $p_0 \in P$ and f is a Lusin function on X with values in a topological space. Then for U, P, p_0, φ and f the conditions (M), (R) and (T) are satisfied. If, moreover, Y is a uniform space then (L) and (S) are also satisfied.

Proof. Let us first prove that (M) is satisfied. Let us choose a sequence of compact sets K_i, $i = 1, 2, \ldots$ in X such that $f|K_i$ is continuous and $\nu(X \backslash K_i) \to 0$. Let V be any open subset of Y. Since $(f|K_i)^{-1}(V)$ is relatively open in K_i, it is a Borel subset of X. With the notation $K = \cup_{i=1}^{\infty} K_i$ we see that $B = (f|K)^{-1}(V)$

is a Borel subset of X. The set $E = X\backslash K$ has ν measure zero, hence the set $N = (f|E)^{-1}(V)$ is also a zero set. Now let us observe that

$$(f \circ \varphi_p)^{-1}(V) = \varphi_p^{-1}(B) \cup \varphi_p^{-1}(N).$$

On the left hand side, $\varphi_p^{-1}(B)$ is a Borel set and by condition (1), the set $\varphi_p^{-1}(N)$ has measure zero. This means that (M) is satisfied.

Now we suppose that Y is a uniform space and we will show that (L) is satisfied. Let C be a compact subset of U, and let $K = \varphi_{p_0}(C)$. Let $\varepsilon > 0$ and let us choose a $\delta > 0$ corresponding to $\varepsilon/2$ by (1). Let us choose an open subset V containing K such that $\nu(V\backslash K) < \delta/2$. Since f is a Lusin function, there exists a compact subset K_0 of K such that $\nu(K\backslash K_0) < \delta/2$ and $f|K_0$ is continuous. Let us choose a uniformity on the compact Hausdorff space K_0 compatible with the topology. Since $f|K_0$ is also uniformly continuous, for each reflexive symmetric relation α from the uniformity of Y there exists a reflexive symmetric relation β from the uniformity of K_0 such that $f(x)$ and $f(x')$ are α-near in Y whenever x and x' are β-near in K_0. Let us choose a reflexive symmetric relation γ from the uniformity of K_0 for which $\gamma \circ \gamma \subset \beta$. For each $u \in C$ there exists an open neighborhood $U_u \subset U$ of u and an open neighborhood P_u of p_0 such that $U_u \times P_u$ is mapped by φ into V and each point of $\varphi(U_u \times P_u)$ which is in K_0, is γ-near to $\varphi(u, p_0)$. Choosing a finite subcover $U_{u_1}, U_{u_2}, \ldots, U_{u_n}$ of C, for $P_0 = \cap_{i=1}^n P_{u_i}$ we obtain that for each $p \in P_0$ the mapping φ_p maps C into V and for any $u \in C$, if $\varphi(u,p)$ is in K_0 then it is β-near to $\varphi(u, p_0)$. Let $p \in P_0$ and let us consider the set $C \cap \varphi_p^{-1}(K_0) \cap \varphi_{p_0}^{-1}(K_0)$. This set is mapped into K_0 by φ_p and by φ_{p_0} too, and for any u from it, $\varphi(u,p)$ and $\varphi(u,p_0)$ are β-near in K_0, hence $f(\varphi(u,p))$ and $f(\varphi(u,p_0))$ are α-near in Y. If we prove that the complement of this set has measure less than ε, then we are done. Since the complement of this set with respect to C is covered by the union of $C\backslash\varphi_p^{-1}(K_0)$ and $C\backslash\varphi_{p_0}^{-1}(K_0)$, it is enough to estimate the measure of these sets. The first set is mapped by φ_p into $V\backslash K_0$, hence it cannot have measure greater than or equal to $\varepsilon/2$. The second set is mapped by φ_{p_0} also into $V\backslash K_0$, hence, similarly, it has measure less than $\varepsilon/2$.

In the remaining part of the proof we use the observation that whenever K' is a compact subset of X and $C' = \varphi_{p_0}^{-1}(K')$ has finite μ measure, then for each $\varepsilon > 0$ there exists a neighborhood P_0 of p_0 such that for each $p \in P_0$ we have $\mu(C'\backslash\varphi_p^{-1}(K')) < \varepsilon$. To prove this, let us choose a compact subset C'' of the Borel set C' for which $\mu(C'\backslash C'') < \varepsilon/2$ and let $K'' = \varphi_{p_0}(C'')$. Let us choose an open set V containing K'' such that $\nu(V\backslash K'') < \delta$, where δ corresponds to $\varepsilon/2$ by (1). For each $u \in C''$ there exist open neighborhoods U_u and P_u of u and p_0, respectively, such that $\varphi(U_u \times P_u) \subset V$. Let us choose a finite subcovering $U_{u_1}, \ldots, U_{u_n}$ of the covering U_u, $u \in C''$, and let $P_0 = \cap_{i=1}^n P_{u_i}$. Then for $p \in P_0$ the set $C''\backslash\varphi_p^{-1}(K'')$ is mapped by φ_p into $V\backslash K''$, hence has μ measure less than $\varepsilon/2$. Now since $K'' \subset K'$ and $C'\backslash\varphi_p^{-1}(K') \subset (C'\backslash C'') \cup (C''\backslash\varphi_p^{-1}(K''))$ we obtain that $\mu(C'\backslash\varphi_p^{-1}(K')) < \varepsilon$.

Now let us suppose only that Y is a topological space. We will prove that (R) is satisfied, which implies (T). Let again C be a compact subset of U and $K=\varphi_{p_0}(C)$, moreover let $p_m\to p_0$ be a sequence in P. Let $\varepsilon_i=2^{-i}$ and let $\delta_i>0$ be the corresponding sequence of numbers δ by (1). Let us choose a compact subset $K_1\subset K$ such that $f|K_1$ is continuous and $\nu(K\backslash K_1)<\delta_1$ and let $C_1=\varphi_{p_0}^{-1}(K_1)$. Then $\mu(C\backslash C_1)<\varepsilon_1$. By induction, using the statement of the previous paragraph, we may find a sequence of indices $m_1<m_2<\dots$ such that $\mu(C_1\backslash\varphi_{p_j}^{-1}(K_1))<\varepsilon_{i+1}$ whenever $j\ge m_i$. This implies that $\mu(C_1\backslash\cap_{r=1}^{\infty}\varphi_{p_{m_r}}^{-1}(K_1))<\varepsilon_1$. Now let K_2 be a compact subset of K such that $f|K_2$ is also continuous and $\nu(K\backslash K_2)<\delta_2$. Let $C_2=\varphi_{p_0}^{-1}(K_2)$, then $\mu(C\backslash C_2)<\varepsilon_2$. Let us apply induction again, but using the new subsequence instead of the original sequence. Then we obtain a subsequence such that $\mu(C_2\backslash\cap_{s=1}^{\infty}\varphi_{p_{m_{r_s}}}^{-1}(K_2))<\varepsilon_2$. Continuing this process and taking the diagonal sequence, we arrive at a subsequence p_{m_t} of p_m such that the measure of the set

$$E_i=(C\backslash C_i)\cup\Big(\cup_{t=i}^{\infty}(C_i\backslash\varphi_{p_{m_t}}^{-1}(K_i))\Big)$$

is less than $2\varepsilon_i$. Now let $E=\cap_{k=1}^{\infty}\cup_{i=k}^{\infty}E_i$. Clearly, $\mu(E)=0$. If $u\in C\backslash E$ then there exists a k such that $u\notin E_i$ for $i\ge k$. This means on one hand that $u\notin C\backslash C_i$ for $i\ge k$, that is, $u\in C_i$ for $i\ge k$. This implies that $\varphi_{p_0}(u)\in K_i$ for $i\ge k$, in particular $\varphi_{p_0}(u)\in K_k$. On the other hand, if $i\ge k$ then for each $t\ge i$ we have $u\notin C_i\backslash\varphi_{p_{m_t}}^{-1}(K_i)$. We will apply this only for $i=k$ to obtain that $\varphi_{p_{m_t}}(u)\in K_k$ whenever $t\ge k$. Since $f|K_k$ is continuous, we obtain that $f(\varphi_{p_{m_t}}(u))\to f(\varphi_{p_0}(u))$.

2.7. Theorem. *Let X be an open subset of $\mathbb{R}^n$. If Y is a topological space having countable base then every measurable function $f:X\to Y$ is contained in $\mathcal{R}_n(X,Y)$, $\mathcal{T}_n(X,Y)$ and $\mathcal{M}_n(X,Y)$. If moreover Y is a uniform space then f is contained in $\mathcal{L}_n(X,Y)$ and $\mathcal{S}_n(X,Y)$.*

Proof. By Lusin's theorem, f is a Lusin function. Let $U\subset\mathbb{R}^n$ be open, P an open subset of some Euclidean space, $p_0\in P$, $\varphi:U\times P\to X$ a $\mathcal{C}^1$ function for which each φ_p, $p\in P$ is an embedding. We will apply the previous theorem for φ locally. Let $u_0\in U$ and let us choose a $c>0$ such that $|\det(\varphi'_{p_0}(u_0))|>c$. Choosing a neighborhood U_0 of u_0 having compact closure and P_0 of p_0 having compact closure such that φ_p is one-to-one on U_0 for each $p\in P_0$ and $|\det(\varphi'_p(u))|>c$ whenever $u\in U_0$ and $p\in P_0$, by the transformation formulae of integrals we have for any measurable subset $B\subset U_0$ that

$$\lambda^n(\varphi_p(B))=\int_B\big|\det(\varphi'_p(u))\big|\,d\lambda^n(u)\ge c\lambda^n(B).$$

The inequality $\lambda^n(\varphi_p(B))\ge c\lambda^n(B)$ is also satisfied for nonmeasurable sets B, because otherwise we can find a Borel hull $A\supset\varphi_p(B)$ for which $\lambda^n(A)<c\lambda^n(\varphi_p^{-1}(A))$

for some $p \in P_0$. This is a contradiction, because $\varphi_p^{-1}(A)$ is a Borel set, hence measurable.

Now, the previous theorem can be applied for $\varphi|U_0 \times P_0$. As it was mentioned at the definition of (L), etc., this is enough to prove that (L) [(S), (R), (T), (M)] is satisfied for U, P, p_0, φ, λ^n.

3. The main results

3.1. Theorem. *Let Z, Z_i $(i = 1, 2, \dots, n)$ be topological spaces. Let X_i $(i = 1, 2, \dots, n)$ and X be open subsets of Euclidean spaces and let $Y \subset \mathbb{R}^l$ be open. Let D be an open subset of $X \times Y$. Consider the functions $f : X \to Z$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \ldots \times Z_n \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \dots, n)$. Let $U \subset \mathbb{R}^k$ be open, P be an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ a $\mathcal{C}^1$-function, for which φ_p is an immersion of U into X for all $p \in P$, and suppose that the following conditions hold:*

(1) For each $(x, y) \in D$

$$f(x) = h\left(x, y, f_1\left(g_1(x, y)\right), \dots, f_n\left(g_n(x, y)\right)\right);$$

(2) for each fixed $y \in Y$, h is continuous in the other variables;

(3) the function f_i is in $\mathcal{R}_{k+l}$ on X_i $(i = 1, 2, \dots, n)$;

(4) g_i is $\mathcal{C}^1$ on D $(i = 1, 2, \dots, n)$;

(5) for each $u_0 \in U$ there exists a y_0 such that $(\varphi(u_0, p_0), y_0) \in D$ and the rank of the derivative of

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

at (u_0, y_0) is $k + l$ for each $1 \le i \le n$.

Then condition (R) is satisfied for f, U, P, p_0, φ, λ^k.

Proof. Suppose that $p_m \to p_0$. Let us choose an open neighborhood U_0 of u_0, P_0 of p_0, and Y_0 of y_0 such that $(\varphi(u, p), y)$ is in D whenever $u \in U_0$, $p \in P_0$, $y \in Y_0$, moreover, the rank of the derivative of the mapping $(u, y) \mapsto g_i(\varphi(u, p), y)$ is equal to $k + l$ for all $u \in U_0$, $p \in P_0$, $y \in Y_0$ and for $1 \le i \le n$. This is possible, because D is open, g_i and φ are $\mathcal{C}^1$-functions, the rank is lower semicontinuous and $U \times Y$ has dimension $k + l$, hence the rank cannot increase above $k + l$.

Since the function f_1 is in $\mathcal{R}_{k+l}$, there is a subsequence p_{m_r} of p_m such that except for pairs $(u, y) \in U_0 \times Y_0$ from a set E_1 having λ^{k+l} measure zero we have

$$f_1(g_1(\varphi(u, p_{m_r}), y)) \to f_1(g_1(\varphi(u, p_0), y)).$$

Now using for the subsequence p_{m_r} that f_2 is in $\mathcal{R}_{k+l}$ we obtain a subsequence $p_{m_{r_s}}$ for which, except for pairs $(u, y) \in U_0 \times Y_0$ from a set E_2 having λ^{k+l} measure zero we have

$$f_2(g_2(\varphi(u, p_{m_{r_s}}), y)) \to f_2(g_2(\varphi(u, p_0), y)),$$

etc. Finally, we obtain a subsequence p_{m_t} of p_m such that except for a set $E = \cup_{i=1}^{n} E_i$ of pairs $(u,y) \in U_0 \times Y_0$ having λ^{k+l} measure zero we have

$$f_i(g_i(\varphi(u,p_{m_t}),y)) \to f_i(g_i(\varphi(u,p_0),y))$$

for $i = 1,2,\dots,n$. By Fubini's theorem, for almost all $y \in Y_0$ we have for almost all $u \in U_0$ that $(u,y) \notin E$. Fixing any such y, from the functional equation and from the continuity of h for fixed y we obtain that

$$f(\varphi(u,p_{m_t})) \to f(\varphi(u,p_0)),$$

which is condition (R) with the function $\varphi|U_0 \times P_0$.

Hence we have proved that for each $u_0 \in U$ there is an open neighborhood U_0 of u_0 such that for a subsequence p_{m_t} of p_m and for almost all $u \in U_0$ we have

$$f(\varphi(u,p_{m_t})) \to f(\varphi(u,p_0)).$$

Since U is a Lindelöf space, by the remark in the definition we obtain that (R) is satisfied.

3.2. Example. Let us consider the following example:

$$\sum_{i=0}^{n} a_i(x,y) f(x + g_i(y)) = 0$$

whenever $x \in \mathbb{R}^m$, $y \in \mathbb{R}$. Suppose that the functions $a_i : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous and the functions $g_i : \mathbb{R} \to \mathbb{R}^m$ are in $\mathcal{C}^1$. Introducing the variable $x_j = x + g_j(y)$ instead of x, we obtain

$$f(x_j) = -\sum_{i \neq j} \frac{a_i(x_j - g_j(y), y)}{a_j(x_j - g_j(y), y)} f(x_j - g_j(y) + g_i(y)). \tag{1}$$

To see that condition (5) is satisfied we have to check the rank of the matrix

$$\begin{pmatrix} \dfrac{\partial \varphi_{p_0}^{(1)}}{\partial u_1}(u) & \dots & \dfrac{\partial \varphi_{p_0}^{(1)}}{\partial u_k}(u) & \dfrac{d\, g_i^{(1)}}{d\, y}(y) - \dfrac{d\, g_j^{(1)}}{d\, y}(y) \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial \varphi_{p_0}^{(m)}}{\partial u_1}(u) & \dots & \dfrac{\partial \varphi_{p_0}^{(m)}}{\partial u_k}(u) & \dfrac{d\, g_i^{(m)}}{d\, y}(y) - \dfrac{d\, g_j^{(m)}}{d\, y}(y) \end{pmatrix},$$

where $\varphi_p^{(r)}$ and $g_i^{(r)}$ are the coordinate functions of φ_p and g_i, respectively. If this is $k+1$, then we may apply our theorem with $l = 1$. This means, geometrically, that the vector $g_i'(y) - g_j'(y)$ is not contained in the range of the linear operator

$\varphi'_{p_0}(u)$ (which is known to be k-dimensional). This range can be any k-dimensional linear subspace in $\mathbb{R}^m$. It may happen that for each k-dimensional linear subspace, there exists a $y \in \mathbb{R}$ such that none of the vectors $g'_i(y) - g'_j(y)$, $i \neq j$ is contained in the linear subspace. Then our theorem can be applied directly and proves that $f \in \mathcal{R}_{k+1}$ implies $f \in \mathcal{R}_k$. If this is the case for $k = m-1, m-2, \dots, 0$ then we obtain that every measurable solution is continuous. But there are situations when this is not the case. If, for example, the derivative of the functions g_i is constant, i. e. if $g_i(y) = b_i + yc_i$, then for any fixed j, equation (1) cannot be applied to get $f \in \mathcal{R}_k$ from $f \in \mathcal{R}_{k+1}$, because for some functions φ the range of $\varphi'_{p_0}(u)$ will contain some of the vectors $g'_i(y) - g'_j(y) = c_i - c_j$. But we have the possibility to use *any* of the equations (1). Using that to be in $\mathcal{R}_k$ is a local property, it is enough to prove that for any k-dimensional linear subspace of $\mathbb{R}^m$ there exists a j such that none of the vectors $c_i - c_j$, $i \neq j$ is contained in the given subspace. For example this is the situation if $n \geq m$ and the vectors $c_0, \dots, c_n$ are in general position. If this condition is not satisfied, then it is still possible that our theorem can be applied. A similar (but somewhat simpler) situation was studied in the paper [13], in the proof of Theorem 2.3.

3.3. Remark. Although, as the example above shows, Theorem 3.1 can be applied in several cases, it is not satisfying because condition (5) is too strong. If we want to apply theorem 3.1 to prove that $f \in \mathcal{R}_k$ then φ can be arbitrary. Hence condition (5) implicitly means that the rank of $\dfrac{\partial g_i}{\partial x}$ has to be large, even if $\dfrac{\partial g_i}{\partial y}$ has a large rank. This in practice means that the g_i have to depend on all coordinates of x, which is not comfortable. We want to relax this condition. Instead of supposing that

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

has maximal possible rank $k + l$ at (u_0, y_0) we will only suppose that it has a constant rank k_i (depending on i) on a neighborhood of (u_0, p_0, y_0). But in this case we have to work with functions from $\mathcal{R}_k \cap \mathcal{M}_k$, and, roughly speaking, our theorem says that solutions in $\mathcal{R}_{k+1} \cap \mathcal{M}_{k+1}$ are also in $\mathcal{R}_k \cap \mathcal{M}_k$.

First we deal only with the measurability condition (M). We will use the following lemma to prove that condition (M) for the unknown functions f_i implies condition (M) for f.

3.4. Lemma. *Let X be an open subset of $\mathbb{R}^n$, Y a topological space, $0 \leq k \leq n$ and $f \in \mathcal{M}_k(X, Y)$. If ψ is a $\mathcal{C}^1$ mapping of the open subset U of $\mathbb{R}^m$ into X for which the rank of the derivative is k everywhere, then $f \circ \psi$ is λ^m measurable.*

Proof. The lemma directly follows from the rank theorem. Indeed, the rank theorem implies, that for each $u_0 \in U$ there exists an open neighborhood U_0 such that $\psi|U_0$ can be written as $\alpha \circ p \circ \beta$. Here, with the notation $I = (-1, 1)$, the mapping β is a diffeomorphism of U_0 onto I^m such that $\beta(u_0) = 0$, the projection p of I^m into I^n has the form $p(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_k, 0, \dots, 0)$, and α is a diffeomorphism of I^n onto an open set X_0 mapping 0 into $x_0 = \psi(u_0)$. Identifying the set $I^k \times \{0\} \subset I^n$ with I^k we have that $\alpha|I^k$ is an immersion, hence $\left(f \circ (\alpha|I^k)\right)^{-1}(V)$ is λ^k measurable for each open subset V of Y. Since $p^{-1}(A)$ is λ^m measurable for each λ^k measurable subset A of I^k, and $\beta^{-1}(B)$ is λ^m measurable for each λ^m measurable subset B of I^m, we obtain that $f \circ (\psi|U_0)$ is λ^m measurable. Now using that U is a Lindelöf space, we get the general case.

3.5. Theorem. *Let Z be a topological space and let Z_i $(i = 1, 2, \dots, n)$ be separable metric spaces. Let X_i $(i = 1, 2, \dots, n)$ and X be open subsets of Euclidean spaces and let $Y \subset \mathbb{R}^l$ be open. Let D be an open subset of $X \times Y$. Consider the functions $f : X \to Z$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \ldots \times Z_n \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \dots, n)$. Let $U \subset \mathbb{R}^k$ be open, $\psi : U \to X$ be a $\mathcal{C}^1$ immersion of U into X, and suppose that the following conditions hold:*

(1) For each $(x, y) \in D$

$$f(x) = h\left(x, y, f_1\left(g_1(x, y)\right), \dots, f_n\left(g_n(x, y)\right)\right);$$

(2) for each fixed $y \in Y$, h is continuous in the other variables;

(3) the function f_i is in $\mathcal{M}_{k_i}$ on X_i $(i = 1, 2, \dots, n)$;

(4) g_i is $\mathcal{C}^1$ on D $(i = 1, 2, \dots, n)$;

(5) for each $u_0 \in U$ there exists a y_0 such that $(\psi(u_0), y_0) \in D$ and the rank of the derivative of

$$(u, y) \mapsto g_i(\psi(u), y)$$

is k_i on a neighborhood of (u_0, y_0) for each $1 \le i \le n$.

Then $u \mapsto f(\psi(u))$ is measurable.

Proof. Let us choose an open neighborhood U_0 of u_0 and Y_0 of y_0 such that $(\psi(u), y)$ is in D whenever $u \in U_0$, $y \in Y_0$, moreover, the rank of the derivative of the mapping $(u, y) \mapsto g_i(\psi(u), y)$ is equal to k_i for all $u \in U_0$, $y \in Y_0$ and for $1 \le i \le n$. This is possible by condition (5). By the previous lemma we obtain that the mapping $(u, y) \mapsto f_i(g_i(\psi(u), y))$ is λ^{k+l} measurable. By Fubini's theorem except for a set E_i of points y from Y_0 with λ^l measure zero the mapping $u \mapsto f_i(g_i(\psi(u), y))$ is λ^k measurable on U_0. Hence, except for the set $E = \cup_{i=1}^n E_i$, for all $y \in Y_0$ the mapping

$$u \mapsto (\psi(u), f_1(g_1(\psi(u), y)), \dots, f_n(g_n(\psi(u), y)))$$

of U_0 into $D_y \times Z_1 \times \dots \times Z_n$ is measurable. Since for any fixed y the function h is continuous in other variables, we obtain that for any fixed $y \in Y_0 \backslash E$ the mapping

$$u \mapsto h(\psi(u), y, f_1(g_1(\psi(u), y)), \dots, f_n(g_n(\psi(u), y)))$$

is measurable. This means that $u \mapsto f(\psi(u))$ is measurable on U_0.

Since U is a Lindelöf space, the statement follows.

The following theorem is the key to the generalization 3.7 of theorem 3.1.

3.6. Theorem. *Let $U \subset \mathbb{R}^m$, X and P be open subsets of Euclidean spaces, $p_0 \in P$, Y a separable metric space, $\varphi : U \times P \to X$ a $\mathcal{C}^1$ function, for which* $\operatorname{rank} \varphi'_p(u) = k$ *for each $u \in U$, $p \in P$. If $f \in \mathcal{M}_k(X, Y) \cap \mathcal{L}_k(X, Y)$ then condition (L) is satisfied for f, U, P, p_0, φ and λ^m.*

Proof. Let $u_0 \in U$. Since the rank of $\varphi'_{p_0}(u_0)$ is equal to k, we may write u as $u = (u_1, u_2) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ such that the determinant of

$$\frac{\partial \varphi}{\partial u_1}(u_0, p_0)$$

is not equal to 0. Hence there exists a neighborhood $U_1 \times U_2$ of u_0 and a neighborhood P_0 of p_0 such that the closure $\leq U_1$ of U_1 is compact, $\leq U_1 \times U_2 \subset U$, and the mapping

$$u_1 \mapsto \varphi(u_1, u_2, p)$$

is an immersion of U_1 for each $u_2 \in U_2$, $p \in P_0$. We may suppose that $\lambda^k(U_1)$ and $\lambda^{m-k}(U_2)$ are finite. Since $f \in \mathcal{L}_k$, for each $\varepsilon, \sigma > 0$ and for each $u_2 \in U_2$ there exists a $\delta > 0$ such that if $|u_2' - u_2| < \delta$, $|p - p_0| < \delta$, then $u_2' \in U_2$ and

$$\lambda^k\{u_1 \in U_1 : \operatorname{dist}(f(\varphi(u_1, u_2', p)), f(\varphi(u_1, u_2, p_0))) \geq \sigma/2\} \leq \frac{\varepsilon}{2\lambda^{m-k}(U_2)}.$$

Applying this for $p = p_0$, too, and combining the two inequalities, we obtain that

$$\lambda^k\{u_1 \in U_1 : \operatorname{dist}(f(\varphi(u_1, u_2', p)), f(\varphi(u_1, u_2', p_0))) \geq \sigma\} \leq \frac{\varepsilon}{\lambda^{m-k}(U_2)} \tag{1}$$

for each u_2' for which $|u_2' - u_2| < \delta$ and for each p for which $|p - p_0| < \delta$. For a fixed $\varepsilon, \sigma > 0$, let δ_{u_2} be the δ corresponding to $u_2 \in U_2$.

Let C be an arbitrary compact subset of $U_1 \times U_2$ and let $C_2 = \{u_2 : (u_1, u_2) \in C\}$ be the projection of C. The closed balls with center $u_2 \in C_2$ and radius $< \delta_{u_2}$ gives a Vitali covering of C_2, and hence it is possible to find a disjoint sequence B_i, $i = 1, 2 \dots$ of them which λ^{m-k} almost covers C_2.

Since $f \in \mathcal{M}_k$, by the previous lemma the mappings $u \mapsto f(\varphi(u,p))$ are λ^m measurable for each $p \in P_0$. Hence the mapping

$$u \mapsto \operatorname{dist}(f(\varphi(u,p)), f(\varphi(u,p_0)))$$

is measurable, too, i. e. the sets

$$\{u \in U_1 \times B_i : \operatorname{dist}(f(\varphi(u,p)), f(\varphi(u,p_0))) \geq \sigma\} \tag{2}$$

are λ^m measurable too. Using (1) and Fubini's theorem we obtain that the λ^m measure of the set (2) is at most $\lambda^{m-k}(B_i)\varepsilon/\lambda^{m-k}(U_2)$. Since the sets B_i are a disjoint almost cover of C_2, we have that

$$\lambda^m\{u \in C : \operatorname{dist}(f(\varphi(u,p)), f(\varphi(u,p_0))) \geq \sigma\} \leq \varepsilon.$$

Hence we have proved that each $u_0 \in U$ has a neighborhood $U_0 = U_1 \times U_2$ such that (L) is satisfied on this. By the remark in the definition of (L) the statement follows.

3.7. Theorem. *Let Z be a topological space and let Z_i $(i = 1, 2, \dots, n)$ be separable metric spaces. Let X_i $(i = 1, 2, \dots, n)$ and X be open subsets of Euclidean spaces and $Y \subset \mathbb{R}^l$ be open. Let D be an open subset of $X \times Y$. Consider the functions $f : X \to Z$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \ldots \times Z_n \to Z$, $g_i : D \to X_i$ $(i = 1, 2, \dots, n)$. Let $U \subset \mathbb{R}^k$ be open, P an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ a $\mathcal{C}^1$-function, for which each φ_p, $p \in P$ is an immersion of U into X, and suppose that the following conditions hold:*

(1) For each $(x,y) \in D$

$$f(x) = h\left(x, y, f_1\left(g_1(x,y)\right), \dots, f_n\left(g_n(x,y)\right)\right);$$

(2) for each fixed $y \in Y$, h is continuous in the other variables;
(3) the function f_i is in $\mathcal{R}_{k_i} \cap \mathcal{M}_{k_i}$, $(i = 1, 2, \dots, n)$;
(4) g_i is $\mathcal{C}^1$ on D $(i = 1, 2, \dots, n)$;
(5) for each $u_0 \in U$ there exists a y_0 such that $(\varphi(u_0, p_0), y_0) \in D$ and the rank of the derivative of

$$(u,y) \mapsto g_i(\varphi(u,p), y)$$

is k_i on a neighborhood of the point (u_0, p_0, y_0) for each $1 \leq i \leq n$.

Then the conditions (R) and (M) are satisfied for f, U, P, p_0, φ, λ^k.

Proof. From Theorem 3.5 it follows that condition (M) is satisfied by f, U, P, p_0, φ, λ^k. Let us fix an $u_0 \in U$ and let us choose a y_0 for u_0 by (5). Let us choose open neighborhoods U_0, P_0 and Y_0 of u_0, p_0 and y_0 such that $(\varphi(u,p), y) \in D$ whenever $u \in U_0$, $p \in P_0$ and $y \in Y_0$, moreover the rank of the derivative of

$$(u, y) \mapsto g_i(\varphi(u,p), y)$$

is k_i on $U_0 \times P_0 \times Y_0$ for each $1 \leq i \leq n$. Now the proof that condition (R) is also satisfied is exactly the same as in Theorem 3.1, but we have to use the previous theorem instead of the definition.

4. Further investigation of the new notions

4.1. Conditions. In what follows we will only investigate the situation, where X is a nonvoid open subset of $\mathbb{R}^n$ and f maps X into a separable metric space, because we want to avoid any difficulties arising only from the poor topology of the range Y.

4.2. Remark. There is a kind of locality other than the one treated after Definition 2.2. We have $f \in \mathcal{L}_k(X,Y)$ if and only if each $x_0 \in X$ has an open neighborhood $X_0 \subset X$ such that $f|X_0 \in \mathcal{L}_k(X_0,Y)$. The "only if" part is trivial. To prove the "if" part we will use the notation of Definition 2.2. Let us note that for each point $u_0 \in U$ there exist open neighborhoods U_0 and P_0 of u_0 and p_0, respectively, such that for $x_0 = \varphi(u_0,p_0)$ the set $\varphi(U_0,P_0)$ is contained in X_0. This means that (L) is satisfied for $\varphi|U_0 \times P_0$. Now from the locality principle in the definition we have that $f \in \mathcal{L}_k(X,Y)$. The same locality is true (and the same proof works) for $\mathcal{S}_k$, $\mathcal{R}_k$, $\mathcal{T}_k$ and $\mathcal{M}_k$.

4.3. The class $\mathcal{M}_k$. Let

$$\mathcal{A}_k = \{A \subset X : \xi_A \in \mathcal{M}_k(X, \{0,1\})\}$$

where $\{0,1\}$ is taken as discrete space. It is easy to see that $\mathcal{A}_k$ is a σ-algebra, and a function $f : X \to Y$ is in $\mathcal{M}_k(X,Y)$ if and only if $f^{-1}(V)$ is in $\mathcal{A}_k$ for each open subset V of Y. Hence the investigation of $\mathcal{M}_k(X,Y)$ is reduced to the investigation of the σ-algebra $\mathcal{A}_k$. It is easy to see that $\mathcal{A}_n$ is the class of all λ^n measurable subsets of X and $\mathcal{A}_0$ is the class of all subsets of X. We will prove that *$A \in \mathcal{A}_k$ if and only if for each open set $U \subset \mathbb{R}^k$ and for each immersion $\psi : U \to X$ the set $A \cap \operatorname{rng}\psi$ is χ^k measurable.*

For each $u \in U$, there exists a compact neighborhood C of u such that the restriction of ψ to C is one-to-one. By the transformation formulae of integrals, if $\psi^{-1}(A) \cap C$ is Lebesgue measurable, then $\psi(C) \cap A$ is Hausdorff measurable. In the other direction, if $\psi(C) \cap A$ is Hausdorff measurable, then, using that the

Hausdorff measure of $\psi(C)$ is finite, there exist Borel sets $B, N \subset \psi(C)$ such that $B \subset A$, $(A \cap \psi(C)) \setminus B \subset N$ and $\chi^k(N) = 0$. The sets $(\psi|C)^{-1}(B)$ and $(\psi|C)^{-1}(N)$ are Borel sets, and the later may only have measure 0. This means that the λ^k measure of $(\psi|C)^{-1}(A \setminus B)$ is zero, too, and hence $(\psi|C)^{-1}(A)$ is λ^k measurable.

Now for each $u \in U$ choosing a compact neighborhood C as above, countably many of them covers U. If $A \cap \operatorname{rng} \psi$ is χ^k measurable, then the sets $(\psi|C_i)^{-1}(A)$ are all λ^k measurable, and hence $\psi^{-1}(A)$ is λ^k measurable. In the other direction, if $\psi^{-1}(A)$ is λ^k measurable, then the sets $\psi^{-1}(A) \cap C_i$ are measurable, too, and hence $A \cap \operatorname{rng} \psi = (\cup_i \psi(C_i)) \cap A$ is a χ^k measurable set.

What we have proved until now implies that *every χ^k measurable set is in $\mathcal{A}_k$*, because $\operatorname{rng} \psi$ is always χ^k measurable. *A countably (χ^k, k) rectifiable set is in $\mathcal{A}_k$ if and only if it is χ^k measurable.* We have only to prove that if $A \in \mathcal{A}_k$ is countably (χ^k, k) rectifiable, i. e. if A is χ^k almost subset of a countable union of Lipschitz images of bounded subsets of $\mathbb{R}^k$, then A is χ^k measurable. By Theorem 3.2.29 from [4], $A \subset N \cup (\cup_{i=1}^\infty S_i)$, where $\chi^k(N) = 0$ and each S_i is a k-dimensional $\mathcal{C}^1$ submanifold of X. Dividing S_i into smaller parts, if necessary, we may suppose that each S_i is the image of some open subset of $\mathbb{R}^k$ by a $\mathcal{C}^1$ immersion ψ_i. Since $\psi_i^{-1}(A)$ is λ^k measurable, the set $A \cap \operatorname{rng} \psi_i = A \cap S_i$ is χ^k measurable for each i. Hence

$$A = (A \cap N) \cup (\cup_{i=1}^\infty (A \cap S_i))$$

is χ^k measurable.

There are χ^k nonmeasurable sets in $\mathcal{A}_k$. Any non χ^k measurable subset ([4], 2.2.4) of a purely unrectifiable compact subset with finite χ^k measure is an example. For such a set A, the set $\psi^{-1}(A)$ has measure 0 for each immersion ψ from an open subset of $\mathbb{R}^k$ into X. Example of a purely unrectifiable set can be found in [4], 3.3.20. See moreover [16], 3.17.

4.4. Connections between $\mathcal{M}_k$, $\mathcal{L}_k$, $\mathcal{S}_k$, $\mathcal{R}_k$ and $\mathcal{T}_k$. One of the simplest questions is, whether $f \in \mathcal{M}_k$ implies $f \in \mathcal{L}_k$, $\mathcal{S}_k$, $\mathcal{R}_k$ or $\mathcal{T}_k$. We know that this is true for $k = n$. If $k < n$ then the characteristic function of the intersection of X and an appropriate k-dimensional plane is in $\mathcal{M}_k$ but contained in none of the classes $\mathcal{L}_k$, $\mathcal{S}_k$, $\mathcal{R}_k$, $\mathcal{T}_k$.

In the other direction, suppose, that $f \in \mathcal{L}_k = \mathcal{S}_k \subset \mathcal{R}_k \subset \mathcal{T}_k$. The question is, whether $f \in \mathcal{M}_k$ is satisfied. This is trivial for $k = 0$. We will show that this cannot be proved in ZFC for $0 < k \leq n$. Namely, we will give an example f under the continuum hypothesis for which $f \in \mathcal{L}_k$ but $f \notin \mathcal{M}_k$. By the famous results of Gödel and Cohen, the continuum hypothesis is independent from the axioms of ZFC. This means that $\mathcal{M}_k \subset \mathcal{L}_k$ cannot be proved in ZFC.

Another question is whether $\mathcal{S}_k = \mathcal{R}_k$. This is trivial for $k = 0$. We will show by a counterexample under the continuum hypothesis that for $0 < k < n$ this is not a theorem in ZFC. I do not know anything about the case $k = n$.

Similarly, we may ask whether $\mathcal{R}_k = \mathcal{T}_k$ or at least $\mathcal{M}_k \cap \mathcal{R}_k = \mathcal{M}_k \cap \mathcal{T}_k$. This is also true for $k = 0$. For $0 < k < n$ we will prove that $\mathcal{M}_k \cap \mathcal{R}_k \subsetneqq \mathcal{M}_k \cap \mathcal{T}_k$ hence $\mathcal{R}_k \subsetneqq \mathcal{T}_k$. For $k = n$ we know that $\mathcal{M}_n \subset \mathcal{R}_n \subset \mathcal{T}_n$ hence of course $\mathcal{M}_n \cap \mathcal{R}_n = \mathcal{M}_n \cap \mathcal{T}_n$. I do not know whether $\mathcal{R}_n = \mathcal{T}_n$.

4.5. Hierarchy of function classes belonging to different dimensions. Let us fix dimensions $0 \le k < l \le n$ and let us investigate the connection between the classes $\mathcal{M}_k$, $\mathcal{L}_k$, etc. and classes $\mathcal{M}_l$, $\mathcal{L}_l$, etc.

We may hope that decreasing the dimension conditions (L), (S), etc. become stronger. One of the only two positive results in this direction is that this is true for the conditions (L), (S) and (R) under measurability:

$$\mathcal{M}_k \cap \mathcal{M}_l \cap \mathcal{L}_k \subset \mathcal{L}_l.$$

The proof of this statement is very similar to the proof of Theorem 3.6, therefore we do not repeat the argument.

We will show by a counterexample under the continuum hypothesis that for $k > 0$

$$\mathrm{ZFC} \not\models \mathcal{M}_k \cap \mathcal{L}_k \subset \mathcal{M}_l \cup \mathcal{T}_l.$$

($\models$ indicates that the right hand side is a theorem in the system on the left.)

Similarly we will show by a counterexample under the continuum hypothesis that

$$\mathrm{ZFC} \not\models \mathcal{M}_k \cap \mathcal{L}_k \cap \mathcal{L}_l \subset \mathcal{M}_l$$

except for the trivial case $k = 0$.

It is much easier to see that inclusions in the other direction do not hold in general. Although

$$\mathcal{M}_l \subset \mathcal{M}_0$$

is satisfied trivially, in general

$$\mathcal{M}_l \not\subset \mathcal{M}_k \qquad \text{if} \qquad k > 0.$$

This is shown by the characteristic function of a non χ^k measurable subset of the intersection of X and an appropriate k dimensional plane. The same example shows that

$$\mathcal{M}_l \cap \mathcal{L}_l \not\subset \mathcal{M}_k \cup \mathcal{T}_k.$$

If we take the characteristic function of the intersection of X and an appropriate k dimensional plane, then we see that

$$\mathcal{M}_l \cap \mathcal{L}_l \cap \mathcal{M}_k \not\subset \mathcal{T}_k.$$

We will show that

$$\mathcal{M}_l \cap \mathcal{R}_k \subset \mathcal{M}_k.$$

I do not know whether $\mathcal{R}_k$ may be replaced here by $\mathcal{T}_k$ except for the trivial case $k = 0$.

Let us see the proofs.

4.6. Theorem. *Under the conditions of 4.1 for* $0 \le k < l \le n$ *we have* $\mathcal{M}_l \cap \mathcal{R}_k \subset \mathcal{M}_k$.

Proof. This is trivial for $k = 0$. Otherwise, let ψ be an immersion of an open subset $U \subset \mathbb{R}^k$ into X. Let $u_0 \in U$ and let V be an $l-k$ dimensional subspace of $\mathbb{R}^n$ orthogonal to $\operatorname{rng} \psi'(u_0)$. Let $\pi : \mathbb{R}^{l-k} \to V$ be a linear isometry, and let us define φ by $\varphi(u,p) = \psi(u) + \pi(p)$. Then for $p_0 = 0$ we have $\varphi_{p_0} = \psi$. Let us choose open neighborhoods U_0 and P_0 of u_0 and p_0, respectively, such that $\varphi(U_0, P_0) \subset X$ and φ is an immersion of $U_0 \times P_0$ into X. Since $f \in \mathcal{M}_l$, the mapping $(u,p) \mapsto f(\varphi(u,p))$ is λ^l measurable. Hence for λ^{l-k} almost all $p \in P_0$ the mapping $u \mapsto f(\varphi(u,p))$ is λ^k measurable. Let us choose a sequence $p_m \to p_0$ such that each $u \mapsto f(\varphi(u,p_m))$ is measurable. By $f \in \mathcal{R}_k$ it is possible to choose a subsequence p_{m_s} such that

$$f(\varphi(u, p_{m_s})) \to f(\varphi(u, p_0))$$

for λ^k almost all $u \in U_0$. Hence $u \mapsto f(\psi(u))$ is measurable over U_0, i. e. locally. This implies that $f \in \mathcal{M}_k$.

4.7. Counterexample. *Under the conditions of 4.1 we will show by a counterexample that for* $0 < k < n$ *we have* $\mathcal{M}_k \cap \mathcal{R}_k \subsetneqq \mathcal{M}_k \cap \mathcal{T}_k$.

Proof. For simplicity, we will work with a nonvoid k-dimensional plane in X having the form $V = X \cap W$ where $W = \{(x_1, x_2, \dots, x_k, x^0_{k+1}, \dots, x^0_n) \in \mathbb{R}^n\}$ for some fixed $x^0_{k+1}, \dots, x^0_n$. Without loss of generality we may suppose that $x^0_{k+1} = \dots = x^0_n = 0$. Our function f will depend only on $x_1, \dots, x_k$ and on the distance $r = \sqrt{x^2_{k+1} + \cdots x^2_n}$ from the subspace W. Let $f(x) = 0$ whenever $r = 0$. Let $g(y)$ be 0 or 1 on $\mathbb{R}^k$ depending whether the sum of the integer parts of the coordinates of $y \in \mathbb{R}^k$ is even or odd, respectively. We will use a smoothing h of this "chessboard" function g to define f. The continuous function h is obtained taking the mean of g for a brick around y, namely, on the set of all $z \in \mathbb{R}^k$ for which the difference $z_i - y_i$ of all coordinates is between $-1/4$ and $1/4$. Now for any nonnegative integer m if $r = \alpha 2^{-m} + (1-\alpha)2^{-m-1}$ for some $0 < \alpha \le 1$ then let us define

$$f(y, x_{k+1}, \dots, x_n) = \alpha h(2^m y) + (1-\alpha) h(2^{m+1} y).$$

For $r > 1$ let

$$f(y, x_{k+1}, \dots, x_n) = h(y).$$

Since f is continuous on the two parts V and $X \backslash V$ of X, it is a Borel function, hence it is in $\mathcal{M}_m$ for any $0 \le m \le n$.

First we will prove that $f \notin \mathcal{R}_k$. Let π be the embedding

$$y \mapsto (y_1, \dots, y_k, 0, \dots, 0)$$

of $\mathbb{R}^k$ into $\mathbb{R}^n$. Let us choose a $K \in \mathbb{N}$ and a vector y^0 from $2^{-K}\mathbb{Z}^k$ such that if U is the set of all points y for which all coordinates of $y - y^0$ are greater than zero and less than 2^{-K}, then the closure of $\pi(U)$ is in V. For $p \in \mathbb{R}$ let $\varphi(u,p) = \pi(u) + pe_n$ where e_n is the unit vector $(0, \dots, 0, 1) \in \mathbb{R}^n$. For an appropriate M we have $\varphi(u,p) \in X$ whenever $u \in U$ and $p \in P = \{p : |p| < 2^{-M}\}$. Let $p_0 = 0$ and $p_m = 2^{-m}$ whenever $m > M$. For any subsequence p_{m_s} of p_m it holds that if for a given $u \in U$ for infinitely many s we have $f(\varphi(u, p_{m_s})) = 1$ then $f(\varphi(u, p_{m_s})) \not\to f(\varphi(u, p_0)) = 0$. Hence with the notation $U_m = \{u \in U : f(\varphi(u, p_m)) = 1\}$ convergence can occur only if there exists an S such that for each $s \geq S$ we have $u \notin U_{m_s}$, i. e. if $u \notin \cap_{S=1}^{\infty} \cup_{s=S}^{\infty} U_{m_s}$. Hence convergence almost everywhere may happen only if

$$\lambda^k \left(\cap_{S=1}^{\infty} \cup_{s=S}^{\infty} U_{m_s}\right) = 0.$$

This means that for convergence almost everywhere $\lambda^k(U_{m_s}) \to 0$ is necessary. But this does not hold because $\lambda^k(U_{m_s}) = \lambda^k(U)/2^k$ whenever $m_s > K$.

It is much harder to prove that $f \in \mathcal{T}_k$. Let U be an open subset of $\mathbb{R}^k$, let P be an open subset of some Euclidean space, $p_0 \in P$ and $\varphi : U \times P \to X$ a $\mathcal{C}^1$ function for which each φ_p, $p \in P$ is an immersion. Let $p_m \to p_0$ be a convergent sequence in P. Since the function f is continuous on $X \setminus V$, if $\varphi(u, p_0) \notin V$ then $f(\varphi(u, p_m)) \to f(\varphi(u, p_0))$. Hence we have to deal only with the set $Z = \{u \in U : \varphi(u, p_0) \in V\}$. Let us introduce the notation $U_m^{\varepsilon} = \{u \in Z : f(\varphi(u, p_m)) \geq \varepsilon\}$. We have to prove that for almost all $u \in Z$ there exists a subsequence p_{m_s} of p_m for which $f(\varphi(u, p_{m_k})) \to f(\varphi(u, p_0)) = 0$. This means that for each $\varepsilon > 0$ and for each M there exists an $m \geq M$ such that $u \notin U_m^{\varepsilon}$, i. e., that $u \notin \cup_{\varepsilon > 0} \cup_{M=1}^{\infty} \cap_{m=M}^{\infty} U_m^{\varepsilon}$. Hence we have to prove that this set has λ^k measure zero. Since decreasing ε the set $\cup_{M=1}^{\infty} \cap_{m=M}^{\infty} U_m^{\varepsilon}$ increases, if we take a sequence $\varepsilon_s > 0$ tending to 0 and restrict the union for only these numbers ε_s, the union does not change. Hence it is enough to prove that for each $\varepsilon > 0$ the set $\cup_{M=1}^{\infty} \cap_{m=M}^{\infty} U_m^{\varepsilon}$ has measure zero, or, equivalently, that for each $\varepsilon > 0$ and for each M the set $\cap_{m=M}^{\infty} U_m^{\varepsilon}$ has λ^k measure zero. If this is not the case, then there exists an $\varepsilon > 0$ and an M for which there exists a density point u_0 of this set. Suppose for contradiction that this is the case and let us fix ε, M and u_0. Moreover, we may suppose that $u_0 \in \cap_{m=M}^{\infty} U_m^{\varepsilon}$.

Let us write $\varphi = (\varphi_1, \varphi_2)$ where $\varphi_1(u,p)$ is the first k coordinates of $\varphi(u,p)$ and $\varphi_2(u,p)$ is the last $n-k$ ones. Since u_0 is a density point of Z, too, we have $\varphi'_{2,p_0}(u_0) = 0$ and $\det \varphi'_{1,p_0}(u_0) \neq 0$. Using the proof of the inverse function theorem, it is possible to find a $c > 0$, an open ball U_0 with center u_0 and a neighborhood P_0 of p_0 such that whenever $\mathbb{B}_\delta(u_0)$ is contained in U_0 and $p \in P_0$ then $\mathbb{B}_{c\delta}(\varphi_{1,p}(u_0))$ is contained in $\varphi_{1,p}(\mathbb{B}_\delta(u_0))$. Furthermore we may suppose that $\|\varphi'_{2,p}(u)\| \leq c/(16\sqrt{k})$ whenever $(u,p) \in U_0 \times P_0$. Shrinking U_0 and P_0, if necessary, we may also suppose that for some positive constant C we have $J(\varphi_{1,p})(u) \le C$ whenever $(u,p) \in U_0 \times P_0$, where J is the absolute value of the Jacobian.

Let $\alpha(k)$ denote the λ^k measure of balls having radius 1 in $\mathbb{R}^k$. Then, of course, the λ^k measure of any ball having radius δ is $\alpha(k)\delta^k$. Since u_0 is a density point, there exists a $\delta_0 > 0$ such that for the closed ball $\mathbb{B}_\delta(u_0)$ we have

$$\lambda^k \left(\mathbb{B}_\delta(u_0)\backslash(\cap_{m=M}^{\infty} U_m^\varepsilon)\right) < \frac{c^k\delta^k}{Ck^{k/2}2^{3k}}$$

whenever $0 < \delta \le \delta_0$. For this δ_0 let us choose an $s_0 > 1$ for which $2^{-s_0+1} \le c\delta_0/\sqrt{k}$. Let us choose an M_0 such that for $m \ge M_0$ we have $p_m \in P_0$ and the distance of $\varphi(u_0,p_m)$ from W is less than 2^{-s_0-2}. Let us fix an $m \ge \max\{M, M_0\}$. Since $u_0 \in U_m^\varepsilon$, the distance of $\varphi(u_0,p_m)$ from W is greater than 0 but less than 2^{-s_0-2}. Let us choose an s such that this distance is not less than $3 \cdot 2^{-s-3}$ but less than $3 \cdot 2^{-s-2}$. Clearly $s \ge s_0$. Let $\sqrt{k}2^{-s}/c < \delta \le \sqrt{k}2^{-s+1}/c$. Then we have $0 < \delta \le \delta_0$. Let S denote the set of all those $y \in \mathbb{R}^k$ for which all coordinates of $2^s y$ has the same integer part as the corresponding coordinate of $2^s y_0$ where $y_0 = \varphi_1(u_0,p_m)$. The set S is the cartesian product of intervals having length 2^{-s}. Hence the diameter of S is $\sqrt{k}2^{-s}$ and because $y_0 \in S$, the set S is contained in $\varphi_{1,p_m}(\mathbb{B}_\delta(u_0))$. Using the estimate of $\|\varphi'_{2,p_m}(u)\|$ valid for all $u \in \mathbb{B}_\delta(u_0)$ we obtain the estimate

$$|\varphi_2(u,p_m) - \varphi_2(u_0,p_m)| \le c\delta/(16\sqrt{k}) \le 2^{-s-3}.$$

This implies that the distance of $\varphi(u,p_m)$ from W is between 2^{-s-2} and 2^{-s}. Let S_0 denote those points y of S for which all of the three functions $h(2^s y)$, $h(2^{s+1}y)$ and $h(2^{s+2}y)$ take the value zero. A $y \in S$ is in S_0 if and only if the fractional part of all the coordinates of $2^s y$, $2^{s+1}y$ and $2^{s+2}y$ is between $1/4$ and $3/4$. This means that the fractional part of all the coordinates of $2^s y$ is in $[5/16, 6/16] \cup [10/16, 11/16]$. Hence the λ^k measure of S_0 is 2^{-sk-3k}. If $u \in \mathbb{B}_\delta(u_0)$ and $y = \varphi_1(u,p_m) \in S_0$, then $u \notin \backslash U_m^\varepsilon$. But $J(\varphi_{1,p_m})(u) \le C$, hence by the transformation formulae of integrals we have

$$\lambda^k(\mathbb{B}_\delta(u_0)\backslash U_m^\varepsilon) \ge \frac{2^{-sk-3k}}{C} \ge \frac{c^k\delta^k}{Ck^{k/2}2^{3k}}.$$

This contradicts the choice of δ_0. This contradiction proves that $f \in \mathcal{T}_k$.

For the following counterexamples we need a lemma. The counter examples are related to the existence of the so-called almost invariant sets. These sets were used by Kakutani and Oxtoby to prove that the Lebesgue measure on the complex unit circle can be extended to an invariant measure such that the Hilbert space dimension of the corresponding $\mathbf{L}^2$ space becomes $2^{\mathbf{c}}$, where $\mathbf{c}$ is the cardinal number continuum. The construction below is a refinement of the construction from the paper [6] of the author, where the result of Kakutani and Oxtoby was extended

— among others — to arbitrary locally compact groups. The ideas there are combined with the well-known ideas of Sierpinski to construct under the continuum hypothesis a subset of the unit square with outer measure 1 and containing at most two points on each line. To understand the typical application of this abstract set theoretic lemma, we may think of the case when X is the plane, T is the class of all diffeomorphisms mapping some open subset of the plane onto some other open subset of the plane, $\mathcal{F}$ is the class of all compact plane sets having positive Lebesgue measure, $\mathcal{G}$ is the class of all one-dimensional $\mathcal{C}^1$ submanifolds of the plane and $\mathbf{n} = \mathbf{c} = \aleph_1$.

4.8. Lemma. *Let X be a set and T a class of one-to-one transformations each mapping a subset of X into X and let $\mathcal{F}$, $\mathcal{G}$ be classes of subsets of X. Suppose that there exists a cardinal number $\mathbf{n} > \aleph_0$ with the following properties:*

(1) $\operatorname{card}(X) = \mathbf{n}$*;*

(2) $\operatorname{card}(T) \le \mathbf{n}$*;*

(3) $\operatorname{card}(\mathcal{F}) \le \mathbf{n}$ *and for every $F \in \mathcal{F}$ we have* $\operatorname{card}(F) = \mathbf{n}$*;*

(4) $\operatorname{card}(\mathcal{G}) \le \mathbf{n}$ *and for every $F \in \mathcal{F}$ and $\mathcal{G}_0 \subset \mathcal{G}$ for which* $\operatorname{card}(\mathcal{G}_0) < \mathbf{n}$ *we have* $\operatorname{card}(F \setminus \cup \mathcal{G}_0) = \mathbf{n}$*;*

(5) The class $\mathcal{G}$ is T invariant, i. e. if $G \in \mathcal{G}$, $\tau \in T$ then $\tau(G) \in \mathcal{G}$ and $\tau^{-1}(G) \in \mathcal{G}$.

Then there exists a family $\{X_\gamma\}_{\gamma\in\Gamma}$ of subsets X_γ of X with the following properties:

(6) $\operatorname{card}(\Gamma) = \mathbf{n}$*;*

(7) the sets X_γ, $\gamma \in \Gamma$ are pairwise disjoint;

(8) for each $\gamma \in \Gamma$ and $G \in \mathcal{G}$ we have $\operatorname{card}(X_\gamma \cap G) < \mathbf{n}$*;*

(9) $\operatorname{card}(F \cap X_\gamma) = \mathbf{n}$ *whenever $\gamma \in \Gamma$ and $F \in \mathcal{F}$;*

(10) for every subset Γ_0 of Γ and for every $\tau \in T$

$$\operatorname{card}\Big(\tau(\cup_{\gamma\in\Gamma_0} X_\gamma) \triangle (\tau(X) \cap (\cup_{\gamma\in\Gamma_0} X_\gamma))\Big) < \mathbf{n}.$$

Proof. Let Ω be the smallest ordinal having cardinality $\mathbf{n}$. We may suppose that $\mathcal{F}$ is nonvoid, because otherwise we may replace it with $\{X\}$. Let Y be an arbitrary set with cardinality $\mathbf{n}$. Since $\operatorname{card}(Y \times \mathcal{F}) = \mathbf{n}$, there exists a one-to-one mapping $\alpha \mapsto (y_\alpha, F_\alpha)$ of the set of ordinals $\{\alpha : 0 \le \alpha < \Omega\}$ onto $Y \times \mathcal{F}$. The transfinite sequence $F_0, \dots, F_\alpha, \dots$, $0 \le \alpha < \Omega$ contains every element F of $\mathcal{F}$ exactly $\mathbf{n}$ times. Similarly, we may suppose that $\mathcal{G}$ is nonvoid, because otherwise we may replace it with $\{\emptyset\}$, and we may choose a transfinite sequence $G_0, \dots, G_\alpha, \dots$, $0 \le \alpha < \Omega$ containing all elements of $\mathcal{G}$. Let us choose a mapping $\alpha \mapsto \tau_\alpha$ of the set $\{\alpha : 0 \le \alpha < \Omega\}$ onto the set $\{\mathbf{1}_X\} \cup T$ for which $\tau_0 = \mathbf{1}_X$ where $\mathbf{1}_X$ is the identical mapping of X onto itself. For each $x \in X$ and each ordinal $\alpha < \Omega$ let $C_\alpha(x)$ denote the set of all points of X that can be written as

$$\tau_{\beta_1}^{\varepsilon_1} \circ \cdots \circ \tau_{\beta_n}^{\varepsilon_n}(x)$$

where $n = 1, 2, \ldots,$ $k = 1, 2, \ldots, n$, $0 \le \beta_k \le \alpha$ and ε_k is 1 or -1. Here τ^1 means the mapping τ and τ^{-1} means the inverse of τ. Clearly, we have $x \in C_\alpha(x)$ and for $x \in X$ and $0 \le \beta \le \alpha < \Omega$ we have $C_\beta(x) \subset C_\alpha(x)$ and $\tau_\beta\left(C_\alpha(x)\right) = \tau_\beta(X) \cap C_\alpha(x)$. We also have

$$\operatorname{card}\left(C_\alpha(x)\right) \le \max\{\operatorname{card}(\alpha), \aleph_0\} < \mathbf{n}.$$

If $A \subset X$ then we will use the notation $C_\alpha(A)$ for $\cup_{x\in A} C_\alpha(x)$. We will show that there exists a transfinite double sequence

$$\{x_\beta^\alpha : 0 \le \beta \le \alpha < \Omega\}$$

of elements of X such that:

$$x_\beta^\alpha \in F_\alpha \quad \text{if} \quad 0 \le \beta \le \alpha < \Omega;$$

$$\text{the sets} \quad \{C_\alpha(x_\beta^\alpha) : 0 \le \beta \le \alpha < \Omega\} \quad \text{are pairwise disjoint;}$$

$$C_\alpha(x_\alpha^\beta) \text{ is disjoint from any } C_\alpha(G_\gamma),\ \gamma \le \alpha.$$

If we agree that $(\gamma, \delta) < (\alpha, \beta)$ whenever $\gamma < \alpha$ or $\gamma = \alpha$ and $\delta < \beta$ (lexicographic ordering), then $\{(\alpha, \beta) : 0 \le \beta \le \alpha < \Omega\}$ is a well ordered set. We will define the sequence $\{x_\beta^\alpha : 0 \le \beta \le \alpha < \Omega\}$ by transfinite induction. Let x_0^0 be an arbitrary point of $F_0 \backslash G_0$. Suppose that $0 \le \beta \le \alpha < \Omega$ and that x_δ^γ have already been defined for all pairs $(\gamma, \delta) < (\alpha, \beta)$, $0 \le \delta \le \gamma$. Consider the union $D(\alpha, \beta)$ of the sets $C_\alpha(x_\delta^\gamma)$ as (γ, δ) runs over all pairs $(\gamma, \delta) < (\alpha, \beta)$. Then

$$\operatorname{card}\left(D(\alpha, \beta)\right) \le (\operatorname{card}(\alpha))^2 \max\{\operatorname{card}(\alpha), \aleph_0\} < \mathbf{n}.$$

Let $E(\alpha)$ be the union of all sets $C_\alpha(G_\gamma)$, $\gamma \le \alpha$. By (5), $E(\alpha)$ is the union of some $\mathcal{G}_\alpha \subset \mathcal{G}$ with $\operatorname{card}(\mathcal{G}_\alpha) < \mathbf{n}$. By (4) the cardinal number of $F_\alpha \backslash E(\alpha)$ is $\mathbf{n}$, hence $(F_\alpha \backslash E(\alpha)) \backslash D(\alpha, \beta)$ is nonvoid. Let x_β^α be an arbitrary point of $(F_\alpha \backslash E(\alpha)) \backslash D(\alpha, \beta)$. Then $C_\alpha(x_\beta^\alpha)$ is disjoint from every $C_\zeta(x)$ where $x = x_\delta^\gamma$ for some $(\gamma, \delta) < (\alpha, \beta)$ or $x \in G_\zeta$ for some $\zeta \le \alpha$. Otherwise we would have

$$\tau_{\beta_1}^{\varepsilon_1} \circ \cdots \circ \tau_{\beta_n}^{\varepsilon_n}(x_\beta^\alpha) = \tau_{\delta_1}^{\eta_1} \circ \cdots \circ \tau_{\delta_m}^{\eta_m}(x),$$

where $\beta_k \le \alpha$, $\delta_j \le \alpha$, ε_k is 1 or -1, and η_j is 1 or -1, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Hence

$$x_\beta^\alpha = \tau_{\beta_n}^{-\varepsilon_n} \circ \cdots \circ \tau_{\beta_1}^{-\varepsilon_1} \circ \tau_{\delta_1}^{\eta_1} \circ \cdots \circ \tau_{\delta_m}^{\eta_m}(x),$$

and this contradicts the choice of x_β^α.

Now let

$$\Gamma = \{\zeta : \ \zeta \text{ is an ordinal and } 0 \le \zeta < \Omega\};$$

$$X_\zeta = \bigcup \left\{ C_\alpha(x_\zeta^\alpha) : \zeta \le \alpha < \Omega \right\}, \qquad \zeta \in \Gamma.$$

Properties (6) and (7) are obvious. Since $x_\zeta^\alpha \in F$ and $x_\zeta^\alpha \in C_\alpha(x_\zeta^\alpha) \subset X_\zeta$ whenever $\zeta \le \alpha < \Omega$ and $F_\alpha = F$, we have that $F \cap X_\zeta$ has at least $\mathbf{n}$ elements. Hence (9) is satisfied.

To prove (8) let us observe that

$$C_\alpha(x_\zeta^\alpha) \cap G_\gamma = \emptyset$$

whenever $\alpha \ge \gamma$. Hence, if $G = G_\gamma$ then

$$X_\zeta \cap G \subset \cup\{C_\alpha(x_\zeta^\alpha) : \zeta \le \alpha < \gamma\}$$

and the right hand side has cardinality less than $\mathbf{n}$.

To prove (10) let $\Gamma_0 \subset \Gamma$ and $\tau \in T$. Suppose that $0 \le \gamma < \Omega$ and $\tau_\gamma = \tau$. Using that

$$\tau_\gamma\left(C_\alpha(x_\zeta^\alpha)\right) = \tau_\gamma(X) \cap C_\alpha(x_\zeta^\alpha) \quad \text{if} \quad \gamma \le \alpha < \Omega$$

and

$$\bigcup_{\zeta \in \Gamma_0} X_\zeta = \bigcup \left\{ C_\alpha(x_\zeta^\alpha) : \zeta \in \Gamma_0, \quad \zeta \le \alpha < \Omega \right\},$$

we have that

$$\begin{aligned} &\tau_\gamma(\cup_{\zeta\in\Gamma_0} X_\zeta) \triangle \big(\tau_\gamma(X) \cap (\cup_{\zeta\in\Gamma_0} X_\zeta)\big) \\ \subset &\bigcup \Big\{ \tau_\gamma\left(C_\alpha(x_\zeta^\alpha)\right) \cup C_\alpha(x_\zeta^\alpha) : \zeta \in \Gamma_0, \quad \zeta \le \alpha < \gamma \Big\}. \end{aligned}$$

Since

$$\operatorname{card}\left(C_\alpha(x_\zeta^\alpha) \cup \tau_\gamma\left(C_\alpha(x_\zeta^\alpha)\right)\right) \le \max\left\{\operatorname{card}(\alpha), \aleph_0\right\},$$

the right hand side has cardinality less than $\mathbf{n}$. Hence (10) is proved.

4.9. Counterexample. *Using the conditions of 4.1, under the continuum hypothesis for* $0 < k \le n$ *we have* $\mathcal{L}_k \not\subset \mathcal{M}_k$.

Proof. We will give a function $f \in \mathcal{L}_k$ for which $f \notin \mathcal{M}_k$. We want to apply the previous lemma. We will use only that the functions φ in the definition of $\mathcal{L}_k$ are continuous and that by Remark 2.3.(3) we may suppose that the functions φ_p are one-to-one. Let T denote the class of all one-to-one functions τ which can be represented in the form $\varphi_p \circ \varphi_{p'}^{-1}$, where U is an open subset of $\mathbb{R}^k$, P is an open subset of some Euclidean space and $\varphi : U \times P \to X$ is a continuous function for which all φ_p, $p \in P$ is one-to-one. Since the cardinality of all pairs U, P is continuum and any continuous function φ is uniquely determined by the values on a countable dense subset, the cardinality of the class T is continuum.

Let $\mathcal{F}$ denote the class of all compact k rectifiable subsets of X having positive χ^k measure. Since each compact set is uniquely determined by its complement, and the open complement is determined by its subsets from a fixed countable base, it follows that the class $\mathcal{F}$ has $\mathbf{c}$ elements, and all elements have cardinality $\mathbf{c}$.

Applying the previous lemma with $\mathcal{G} = \emptyset$ we obtain a class of subsets X_γ, $\gamma \in \mathbb{R}$ of X. Our counterexample will be the characteristic function f of X_0 i.e. X_γ for $\gamma = 0$.

Let U be a bounded open subset of $\mathbb{R}^k$ and $\psi : U \to X$ be an immersion for which the rectifiable and χ^k measurable set $M = \psi(U)$ has positive but finite χ^k measure. Let us observe that if $X_0 \cap M$ were of χ^k measure zero, then $M \backslash X_0$ would contain some $F \in \mathcal{F}$, which is impossible because $F \cap X_0 \neq \emptyset$. If $X_0 \cap M$ were χ^k measurable with positive χ^k measure then it would contain some $F \in \mathcal{F}$. But this is impossible because $F \cap X_\gamma \neq \emptyset$ and $X_\gamma \cap X_0 = \emptyset$ for any $\gamma \neq 0$. Hence $X_0 \cap M$ is non χ^k measurable. By 4.3 this implies that $f \notin \mathcal{M}_k$.

We will prove that $f \in \mathcal{L}_k$. Let C be a compact subset of U. The set

$$\{u \in C : f(\varphi_{p_0}(u)) \neq f(\varphi_p(u))\}$$

is equal to the set

$$\varphi_{p_0}^{-1}\left(\{x \in \varphi_{p_0}(C) : x \in X_0 \triangle (\varphi_{p_0} \circ \varphi_p^{-1})(X_0)\}\right).$$

For the mapping $\tau = \varphi_{p_0} \circ \varphi_p^{-1}$ this set is a subset of the set

$$\varphi_{p_0}^{-1}\left((\tau(X) \cap X_0) \triangle \tau(X_0)\right).$$

If we suppose that the continuum hypothesis holds then this set is countable.

4.10. Counterexample. *Using the conditions of 4.1, for $0 < k < n$ under the continuum hypothesis $\mathcal{S}_k \subsetneqq \mathcal{R}_k$.*

Proof. We apply the construction of the previous lemma, choosing for T, $\mathcal{F}$ and $\mathcal{G}$ the same classes as above to obtain the sets X_γ, $\gamma \in \mathbb{R}$. If $m \in \mathbb{N}$ and $m \geq 2$, let $f_m(x) = g_m(x) h_m(x)$, where $g_m(x)$ is the characteristic function of the set X_m, and

$$h_m(x) = \begin{cases} 0, & \text{if } \operatorname{dist}(x, D) \leq \frac{1}{m+1}; \\ 0, & \text{if } \operatorname{dist}(x, D) \geq \frac{1}{m-1}; \\ m(m+1)\left(\operatorname{dist}(x, D) - \frac{1}{m+1}\right) & \text{if } \frac{1}{m+1} \leq \operatorname{dist}(x, D) \leq \frac{1}{m}; \\ m(m-1)\left(\frac{1}{m-1} - \operatorname{dist}(x, D)\right) & \text{if } \frac{1}{m} \leq \operatorname{dist}(x, D) \leq \frac{1}{m-1}, \end{cases}$$

where D is a given nonvoid k-dimensional closed disk contained in the intersection of X with a k dimensional plane. Let $f = \sum_{m=1}^{\infty} f_m$. (As in the previous counterexample we can prove that $f \notin \mathcal{M}_k$.) As in the previous counterexample it follows that each g_m is in $\mathcal{L}_k = \mathcal{S}_k$, hence in $\mathcal{R}_k$, too. The same is trivial for the continuous function h_m. From this it follows for the product $g_m h_m$ that it is also in $\mathcal{R}_k$. Since everywhere on the open set $X \backslash D$ the function f is locally the finite sum of such products, we have that $f|X \setminus D \in \mathcal{R}_k$. Let $\varphi : U \times P \to X$ and let

$$F = \{u : \varphi(u, p_0) \in D\}.$$

Clearly F is a closed set. Let C be a compact subset of F. For $p_m \to p_0$, let $R_{m,j}$ denote the set of all points u for which $\varphi(u, p_m) \in X_j$ but $\varphi(u, p_0) \notin X_j$ or $\varphi(u, p_m) \notin X_j$ but $\varphi(u, p_0) \in X_j$. Under the continuum hypothesis, the sets $R_{m,j}$ and their union $R = \cup_{m,j=1}^{\infty} R_{m,j}$ are countable and hence have λ^k measure zero.

Let us observe that for each i there exists an m_i such that if $m > m_i$ then for each $u \in C$ we have

$$|\varphi(u, p_m) - \varphi(u, p_0)| < \frac{1}{i+1}.$$

Hence, if $u \in C$ but $u \notin R$ and $u \notin \cup_{j=i}^{\infty} \varphi_{p_0}^{-1}(X_j)$, then $\varphi(u, p_m) \notin X_j$ whenever $j \geq i$. Hence $g_j(\varphi(u, p_m)) = 0$ for $j \geq i$. On the other hand, $\mathrm{dist}(\varphi(u, p_m), D) < \frac{1}{i+1}$, hence $h_j(\varphi(u, p_m)) = 0$ whenever $j \leq i$. So we obtain that $f(\varphi(u, p_m)) = 0$ whenever $u \notin R$, $u \notin \cup_{j=i}^{\infty} \varphi_{p_0}^{-1}(X_j)$ and $m > m_i$. Since the sets X_j are disjoint, $f(\varphi(u, p_m)) \to f(\varphi(u, p_0))$ for $m \to \infty$ whenever $u \notin R$, i. e. almost everywhere. Taking union for countably many sets C we obtain that $f \in \mathcal{R}_k$.

On the other hand, if $e \neq 0$ is orthogonal to D and $\varphi(u, p) = \psi(u) + pe$, where ψ is an isometric immersion mapping some nonvoid open subset of $\mathbb{R}^k$ into D, $p_0 = 0$, then, for $p_m = 1/m$ we have that

$$\{u \in C : |f(\varphi(u, p_m)) - f(\varphi(u, p_0))| \geq 1\} \supset \psi^{-1}(X_m) \cap C,$$

if m is large enough, except for a countable set. The set on the left hand side has the same λ^k measure as C. This shows that $f \notin \mathcal{S}_k$.

4.11. Counterexample. *Using the conditions of 4.1, under the continuum hypothesis for* $0 < k < l \leq n$ *we have* $\mathcal{M}_k \cap \mathcal{L}_k \cap \mathcal{L}_l \not\subset \mathcal{M}_l$.

Proof. We will give an example of a function $f \in \mathcal{M}_k \cap \mathcal{L}_k \cap \mathcal{L}_l$ but $f \notin \mathcal{M}_l$. We want to apply Lemma 4.8. We will use that by Remark 2.3.(3) we may suppose that the functions φ_p in the definition of $\mathcal{L}_l$ are one-to-one immersions. Let T denote the class of all one-to-one functions τ which can be represented in the form $\varphi_p \circ \varphi_{p'}^{-1}$, where U is an open subset of $\mathbb{R}^l$, P is an open subset of some Euclidean space and $\varphi : U \times P \to X$ is a $\mathcal{C}^1$ function for which all φ_p, $p \in P$ are one-to-one.

Let $\mathcal{F}$ denote the class of all compact l rectifiable subsets of X having positive χ^l measure. Let $\mathcal{G}$ be the class of all k-rectifiable Borel subsets of X. It is not hard to prove that the class $\mathcal{G}$ is T invariant. Moreover all $G \in \mathcal{G}$ has χ^l measure zero, hence the same is true for the union of countably many $G \in \mathcal{G}$. This means that $F \setminus \cup \mathcal{G}_0$ has positive χ^l measure, hence cardinality $\mathbf{c}$ for any countable subfamily $\mathcal{G}_0 \subset \mathcal{G}$ and for any $F \in \mathcal{F}$. Other conditions of Lemma 4.8 has already been checked at 4.9.

Applying Lemma 4.8 we obtain a class X_γ, $\gamma \in \mathbb{R}$ where each X_γ contains only countably many points from each $G \in \mathcal{G}$, but $X_\gamma \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, hence $X_\gamma \cap F$ is not χ^l measurable for any $F \in \mathcal{F}$.

Let f be the characteristic function of X_0. Along the same lines as in 4.9 we get that $f \in \mathcal{L}_l$ but $f \notin \mathcal{M}_l$. Since for any $\mathcal{C}^1$ embedding ψ of an open subset of $\mathbb{R}^k$ into X the function $f \circ \psi$ is zero except for a countable set, we get that $f \in \mathcal{M}_k$ and $f \in \mathcal{L}_k$, too. Hence the statement is proved.

4.12. Counterexample. *Using the conditions of 4.1, under the continuum hypothesis for* $0 < k < l \leq n$ *we have* $\mathcal{M}_k \cap \mathcal{L}_k \not\subset \mathcal{M}_l \cup \mathcal{T}_l$.

Proof. Let us apply Lemma 4.8 for the same T, $\mathcal{F}$ and $\mathcal{G}$ as in the previous counterexample. We obtain a class X_γ, $\gamma \in \mathbb{R}$ where each X_γ contains only countably many points from each $G \in \mathcal{G}$, but $X_\gamma \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, hence $X_\gamma \cap F$ not χ^l measurable for any $F \in \mathcal{F}$.

Let Z be an l dimensional plane which has a nonempty intersection with X and let f be the characteristic function of the set $Z \cap X_0$. Then $f \in \mathcal{M}_k \cap \mathcal{S}_k = \mathcal{M}_k \cap \mathcal{L}_k$, but $f \notin \mathcal{T}_l$ and $f \notin \mathcal{M}_l$.

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