# CONTINUITY IMPLIES DIFFERENTIABILITY FOR SOLUTIONS OF FUNCTIONAL EQUATIONS — EVEN WITH FEW VARIABLES

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Abstract. It is proved that — under certain conditions — continuous solutions $f$ of the functional equation

$$f(x) = h(x, y, f(g_1(x, y)), \dots , f(g_n(x, y))), \quad (x, y) \in D \subset \mathbb{R}^s \times \mathbb{R}^l,$$

are $\mathcal{C}^\infty$, even if $1 \le l \le s$. As a tool we introduce new function classes which — roughly speaking — interpolate between differentiable and continuous functions.

## 1. Introduction

In connection with his fifth problem Hilbert [4] suggested that although the method of reduction to differential equations makes it possible to solve functional equations in an elegant way, the inherent differentiability assumptions are typically unnatural (see [2]). Such shortcomings can be overcome by applying regularity theorems.

In this spirit the following general regularity problem for non-composite functional equations with several variables was formulated (in a somewhat different form) by the author and included by Aczél among the most important open problems on functional equations (see Aczél [1] and Járai [5]):

**1.1. Problem.** *Let $X$, $Y$ and $Z$ be open subsets of $\mathbb{R}^s$, $\mathbb{R}^t$ and $\mathbb{R}^m$, respectively, and let $D$ be an open subset of $X \times Y$. Let $f : X \to Z$, $g_i : D \to X$ $(i = 1, 2, \dots , n)$ and $h : D \times Z^n \to Z$ be functions. Suppose that*

*(1)*
$$f(x) = h(x, y, f(g_1(x, y)), \dots , f(g_n(x, y))) \text{ whenever } (x, y) \in D;$$

*(2) $h$ is analytic;*

*(3) $g_i$ is analytic and for each $x \in X$ there exists a $y$ for which $(x, y) \in D$ and $\dfrac{\partial g_i}{\partial y}(x, y)$ has rank $s$ $(i = 1, 2, \dots , n)$.*

*Is it true that every $f$ which is measurable or has the Baire property is analytic?*

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The following steps can be used:

(I) Measurability implies continuity.

(II) Baire property implies continuity.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All $p$ times continuously differentiable solutions are $p+1$ times continuously differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

Simple examples show that none of the conditions of the problem can be omitted. Moreover, using a general "transfer principle" several regularity problems concerning functional equations with more than one unknown functions can be reduced to the problem above. We note that in order to obtain $f \in \mathcal{C}^p$ it is usually enough to suppose only that the given functions $h$ and $g_i$ are in $\mathcal{C}^p$ (if $2 \le p \le \infty$) or in $\mathcal{C}^{p+1}$ (if $p = 0$ or $p = 1$).

The complete answer to the problem above is not known. The author discussed this problem in several papers and solved problems corresponding to (I), (II), (IV) and (V) (see [5]), and under some additional compactness condition (III) (see [7]). References can be found in the survey paper [14]. There are some partial results in connection with (VI). Moreover, other properties of solutions such as having locally bounded variation or local Hölder continuity are also discussed (see [9] and references in [14]). It is possible to extend these results to manifolds, and the $\mathcal{C}^\infty$-part of the problem is completely solved on compact manifolds [10]. The most applicable results are treated in the booklet [8].

Regularity theorems of the type "locally integrable solutions are infinitely many times differentiable" can be obtained using distributions. The essence of the method is to prove that solutions in the distribution sense satisfy a differential equation having only infinitely many times differentiable solutions. This idea was used by Światak [16] to prove general regularity results for the functional equation

$$\sum_{i=1}^{n} h_i(x,y) f(g_i(x,y)) = h(x, f(g_{n+1}(x)), \dots, f(g_m(x))) + h_0(x,y), \tag{4}$$

where $f$ is the only unknown function. Roughly speaking, she applies a partial differential operator in $y$ to the equation in the distribution sense. Of course, the nonlinear term on the right hand side disappears. If there exists a $y_0$ such that $g_i(x, y_0) \equiv x$ for $1 \le i \le n$, and substituting this $y_0$ we are fortunate enough to obtain a hypoelliptic partial differential equation, then by the regularity theory of partial differential equations all distribution solutions are in $\mathcal{C}^\infty$. For the exact details of how to overcome the difficulties and for applications see her paper [16].

Further references about regularity theorems for functional equations can be found in the survey paper [14]. Some other papers concerning the distribution method are also referred there.

The above equation of Światak is "almost linear", so, formally, it is much less general than equation (1). However, her theorems can be applied in certain cases even if the rank of $\frac{\partial g_i}{\partial y}$ is much smaller than the dimension of the domain of the unknown function $f$. Roughly speaking, the present author's results, quoted above, may be applied to prove regularity of a solution $f$ having $m$ variables, only if there are at least $2m$ variables in the functional equation. The method of Światak may apply even if there are only $m+1$ variables. This is the minimal number of variables: in Hilbert's paper [4] there is an example that for "one variable" functional equations (this may mean an $m$-dimensional vector variable) no regularity theorem holds. So the results of Światak suggest that the rank condition in the problem above is too strong, and the results can be extended to a much more general case.

Such "measurability implies continuity even with few variables" type results were treated recently in Járai [12]. Well-known analogies between measurability and Baire property suggest analogous results for Baire property (see [11]).

In this paper we prove general "continuity implies $\mathcal{C}^\infty$" type results. The "$\mathcal{C}^1$ implies $\mathcal{C}^\infty$" part holds for the general explicit nonlinear functional equation (1) without the strong rank condition in (3) to the inner functions. The "continuity implies $\mathcal{C}^1$" part holds only for linear equations of the type

$$f(x) = h_0(x,y) + \sum_{i=1}^{n} h_i(x,y) f(g_i(x,y)) \tag{5}$$

with unknown function $f$. In the spirit of the "bootstrap" method corresponding to steps (I)–(VI) we introduce a sequence of properties, which — roughly speaking — interpolate between continuity and continuous differentiability. This sequence of properties provides us with a stairway to climb up from continuity to continuous differentiability. First we investigate the basic properties of the new notions. Then a "continuity implies $\mathcal{C}^1$" type theorem will be proved. An example is given how to apply the theorem in nontrivial cases. A refinement of the theorem is also proved. Finally, a "$\mathcal{C}^1$ implies $\mathcal{C}^\infty$" type theorem is proved.

The main advantages of our method compared to Światak's are the following:

- We do not need the very strong condition that there is a $y_0$ such that $g_i(x,y_0) \equiv x$ for $1 \le i \le n$. This condition does not hold for most of the important functional equations.
- The somewhat artificial condition of hypoellipticity is also avoided. Our conditions — besides smoothness conditions for the given functions — are only linear algebraic in nature.
- Seemingly, equation (4) of Światak is nonlinear, and our equation (5) is linear. However, substituting $y = y_0$ in equation (4) the conditions $g_i(x,y_0) \equiv x$ for $1 \le i \le n$ yield

$$\sum_{i=1}^{n} h_i(x,y_0) f(x) = h(x, f(g_{n+1}(x)), \dots, f(g_m(x))) + h_0(x,y_0).$$

Expressing the term $h(x, f(g_{n+1}(x)), \dots, f(g_m(x)))$ from this equation, and substituting back into (4), after division by $\sum_{i=1}^{n} h_i(x, y_0)$ we obtain an equation of the type (5). Hence our methods can be applied to prove "continuity implies $\mathcal{C}^1$" for Światak's equation. We prove "$\mathcal{C}^1$ implies $\mathcal{C}^\infty$" for the most general *nonlinear* functional equation (3).

- Finally, generalizing our methods we may hope to obtain "continuity implies $\mathcal{C}^1$" type results for the most general *nonlinear* functional equation (3). This seems to be impossible using the method of Światak based on Schwartz distributions. The applicability of the distribution method is restricted, because no multiplication among Schwartz distributions is defined. By Schwartz's impossibility theorem, this cannot be done in a satisfying way. It is even more hopeless to substitute distributions into general $\mathcal{C}^\infty$ functions with several variables. The distribution method has to be restricted to functional equations that are not very far from being linear.

## 2. The new notions

**2.2. Definition.** The basic idea is to consider parametric integrals of the form

$$p \mapsto \int_U w(u,p) f(\varphi(u,p))\, d\mu(u), \tag{1}$$

where $f : X \to Y$ is a function mapping the set $X$ into the Banach space $Y$. Such a parametric integral is given by a *parametric integration quintuple* $(P, U, w, \varphi, \mu)$, where $P$ is the parameter space, $U$ is a measure space with measure $\mu$, the function $w$ is a weight function $w : U \times P \to \mathbb{R}$ and the function $\varphi : U \times P \to X$ is considered representing a parametric family $u \mapsto \varphi(u,p)$, $p \in P$ of surfaces in $X$. We may consider a set $\mathcal{P}$ of parametric integration quintuples and to denote by $\mathcal{F}(X, Y, \mathcal{P}; \mathcal{G})$ the class of all such function $f : X \to Y$ for which for all quintuples from $\mathcal{P}$ the parametric integral (1) is in the class of functions $\mathcal{G}$.

For our purposes a somewhat simpler setting will be sufficient. For simplicity, we may suppose that $f$ is a continuous function mapping an open subset $X$ of $\mathbb{R}^n$ into a Banach space $Y$. We shall consider the set $\mathcal{P}_k$ of *all* integration quintuples $(U, P, w, \varphi, \mu)$ for which $U$ is an open subset of $\mathbb{R}^k$, the "parameter space" $P$ is some open subset of some Euclidean space, $\mu$ is the restriction of Lebesgue measure $\lambda^k$ to subsets of $U$, and $w : U \times P \to \mathbb{R}$ and $\varphi : U \times P \to X$ are arbitrary functions satisfying some smoothness conditions. We use weight functions that are at least continuous and have compact support. The functions $\varphi$ are supposed to be at least $\mathcal{C}^1$. Under such conditions the parametric integral

$$p \mapsto \int_U w(u,p) f(\varphi(u,p))\, du \tag{2}$$

exists for each $p \in P$ and is continuous. Here integration is with respect to $k$ dimensional Lebesgue measure.

Earlier results about parametric integrals (see Theorem 2.5 below) show that if $w \in \mathcal{C}^1$ and if $f$ is merely continuous, but $\varphi$ is twice continuously differentiable, this parametric integral will be continuously differentiable for $k = n$. If $k = 0$ then the continuous differentiability of such integrals becomes equivalent to the continuous differentiability of $f$. Roughly speaking, the continuous differentiability of such parametric integrals is a stronger condition on $f$, the smaller $k$ is, and provides us with a stairway to climb up from continuity to continuous differentiability.

For convenience we introduce the following notation. Suppose that $X$ is an open subset of $\mathbb{R}^n$, $Y$ is a Banach space, $0 \le k \le n$ an integer, $\mathcal{W}$ is a class of functions $w$ mapping some product $U \times P$ (depending on $w$) into $\mathbb{R}$, where $U$ is an open subset of $\mathbb{R}^k$ and $P$ is an open subset of some Euclidean space, $\Phi$ is a class of functions $\varphi$ mapping some product $U \times P$ into $X$ where, again $U$ is an open subset of $\mathbb{R}^k$ and $P$ is an open subset of some Euclidean space, and $\mathcal{G}$ is a class of functions mapping some open subset $P$ of some Euclidean space into $Y$. Let

$$\mathcal{F}_k(X, Y, \mathcal{W}, \Phi; \mathcal{G})$$

denote the class of all continuous functions $f : X \to Y$ for which whenever $w \in \mathcal{W}$ and $\varphi \in \Phi$ have the same domain $U \times P$, the parametric integral (2) is defined for all $p \in P$ and is in the class $\mathcal{G}$.

The function classes $\mathcal{W}$, $\Phi$ and $\mathcal{G}$ will be defined via smoothness conditions. Let $0 \le m \le \infty$ and let $\mathcal{C}^m$ denote the class of all functions which are defined on some open subset of some Euclidean space, take values in a Banach space and are $m$ times continuously differentiable. Let $\mathcal{K}^m$ be the subclass of $\mathcal{C}^m$ consisting of functions that have compact support. Let $\mathcal{I}^m$ denote the class of functions $\varphi \in \mathcal{C}^m$ which map some Cartesian product $U \times P$ of open subsets of Euclidean spaces into an Euclidean space so that $u \mapsto \varphi(u, p)$ is an immersion for each $p \in P$. Similarly, let $\mathcal{E}^m$ denote the class of those functions $\varphi \in \mathcal{C}^m$ which are mapping some Cartesian product $U \times P$ of open subsets of Euclidean spaces into an Euclidean space so that $u \mapsto \varphi(u, p)$ is an embedding for each $p \in P$.

The most important classes for us will be $\mathcal{F}_k(X, Y, \mathcal{K}^1, \mathcal{I}^2; \mathcal{C}^1)$, $0 \le k \le n$. We shall often check the condition $f \in \mathcal{F}_k(X, Y, \mathcal{K}^1, \mathcal{I}^2; \mathcal{C}^1)$ locally. If for a given function $\varphi : U \times P \to X$ for each $u_0 \in U$ and $p_0 \in P$ there is a neighbourhood $U_0$ of $u_0$ and $P_0$ of $p_0$ such that the parametric integral (2) is in $\mathcal{C}^1$ whenever the support of $w$ is contained in $U_0 \times P_0$, then for any $w : U \times P \to \mathbb{R}$ the parametric integral (2) is in $\mathcal{C}^1$. This easily follows using a partition of unity.

**2.3. Remarks.** (1) Suppose that $X$ is an open subset of $\mathbb{R}^n$ and let the range be $\mathbb{R}^m$. Our main results will show that, roughly speaking, solutions $f$ from $\mathcal{F}_{k+1}(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{I}^2; \mathcal{C}^1)$ are also in $\mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{I}^2; \mathcal{C}^1)$. We shall prove that the class $\mathcal{F}_0(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{I}^2; \mathcal{C}^1)$ is the class of $\mathcal{C}^1$ functions, and that all continuous functions $f : X \to \mathbb{R}^m$ are in $\mathcal{F}_n(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{I}^2; \mathcal{C}^1)$. Hence, step-by-step, continuity of solutions implies that they are in $\mathcal{C}^1$.

(2) There is some analogy with the measure theoretical and with the Baire category case. About the history of the analogous measure theoretical notions see the references in Járai [12].

(3) It is a classic technique to use parametric integrals to prove regularity theorems for functional equations with unknown functions having one real variable. See the book of Aczél, 4.2.2, 4.2.3 for its history. The generalization for functional equations with unknown functions of several variables is non-trivial and was carried out by the author (see [5]). The main difficulty is to prove that a parametric integral of the form

$$p \mapsto \int_{g_p(D)} h(x,p)\,dx$$

is continuously differentiable; even if we only know that $h$ and $\dfrac{\partial h}{\partial p}$ are continuous and $g$ is continuously differentiable.

(4) If $X$ is an open subset of $\mathbb{R}^n$, $Y$ is a Banach space, and $0 \le k \le n$, then the class $\mathcal{F}_k(X,Y,\mathcal{K}^1,\mathcal{I}^2;\mathcal{C}^1)$ is equal to the class $\mathcal{F}_k(X,Y,\mathcal{K}^1,\mathcal{E}^2;\mathcal{C}^1)$. This easily follows from the locality principle mentioned in the definition.

**2.4. Theorem.** *Let $Y=\mathbb{R}^m$ and let $X$ be an open subset of $\mathbb{R}^n$. Then $\mathcal{F}_0(X,Y,\mathcal{K}^\infty,\mathcal{I}^\infty;\mathcal{C}^k)=\mathcal{C}^k$ for $0 \le k < \infty$.*

**Proof.** In this case the parametric integral simply becomes the mapping $p \mapsto w(0,p)f(\varphi(0,p))$. We may take $P=X$, $p \mapsto \varphi(0,p)$ to be the identity mapping, and $p \mapsto w(0,p)$ to be equal to one in a neighbourhood of a given point $x_0=p_0 \in X$. Then it follows that $f$ is a $\mathcal{C}^k$ function in a neighbourhood of $x_0$. Conversely, if $f \in \mathcal{C}^k$, then the mapping $p \mapsto w(0,p)f(\varphi(0,p))$ is also in $\mathcal{C}^k$.

Next we prove that continuous functions mapping an open subset $X$ of $\mathbb{R}^n$ into $\mathbb{R}^m$ are in $\mathcal{F}_0(X,\mathbb{R}^m,\mathcal{K}^1,\mathcal{I}^2;\mathcal{C}^1)$. To do this we need the following theorem about differentiation of parametric integrals.

**2.5. Theorem.** *Let $S$ be a simplex with non-void interior in $\mathbb{R}^n$, and let $U$, $V$ and $W$ be open subsets of $\mathbb{R}^n$, $\mathbb{R}^s$ and $\mathbb{R}^n$, respectively. Let $g:(t,y)\mapsto x$ be a function mapping $V \times W$ into $U$, and let $h:(t,x)\mapsto z$ be a function mapping $V \times U$ into $\mathbb{R}$. Suppose, that $S \subset W$ and*

*(1) $g$ is continuously differentiable, $g_t$ is invertable for all $t \in V$, and $(Jg_t)(y) \ne 0$ if $t \in V$ and $y \in W$;*

*(2) $h$ and $\dfrac{\partial h}{\partial t}$ are continuous.*

*Then the function*

$$F(t)=\int_{g_t(S)} h(t,x)\,dx \quad \textit{whenever} \quad t \in V$$

*is continuously differentiable on $V$.*

**Proof.** This is the main result of Járai [6]. For our purposes the somewhat weaker version (Theorem 5.1 in Járai [5]) supposing that $g$ is twice continuously differentiable would be also sufficient.

**2.6. Theorem.** *Let $X$ be an open subset of $\mathbb{R}^n$, and let $f : X \to \mathbb{R}^m$ be a continuous function. Then $f \in \mathcal{F}_n(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$.*

**Proof.** Arguing coordinate-wise, we may suppose, that $f$ is real valued, i. e. that $m = 1$. Let $P$ be an open subset of $\mathbb{R}^s$ and let $U$ be an open subset of $\mathbb{R}^n$. Let $\varphi : U \times P \mapsto X$ be a function from $\mathcal{I}^2$. By the locality principle from the definition, it is enough to prove that any $p_0 \in P$ and $u_0 \in U$ have neighbourhoods $P_0 \subset P$ and $U_0 \subset U$, respectively, such that whenever $w : U \times P \to \mathbb{R}$ is a $\mathcal{C}^1$ function having support contained in $U_0 \times P_0$, the parametric integral

$$p \mapsto \int_U w(u,p) f(\varphi(u,p))\, du$$

is in $\mathcal{C}^1$. If $U_0$ and $P_0$ are small enough, then we can substitute $x = \varphi(u,p)$ for each $p \in P_0$. We may choose a simplex $S$ containing $U_0$. Then the integral above became

$$\int_{\varphi_p(S)} w(\varphi_p^{-1}(x), p) f(x) J(\varphi_p^{-1})(x)\, dx$$

for all $p_0 \in P_0$. Hence, by the previous theorem the integral is continuous.

## 3. The main results

**3.1. Theorem.** *Let $X$ and $X_i$, $1 \le i \le n$ be open subsets of Euclidean spaces and let $Y$ be an open subset of $\mathbb{R}^l$. Let $D$ be an open subset of $X \times Y$. Consider the functions $f : X \to \mathbb{R}^m$, $f_i : X_i \to \mathbb{R}^m$, $h_i : D \to \mathbb{R}$, $g_i : D \to X_i$ $(i = 1, 2, \dots, n)$. Let $U \subset \mathbb{R}^k$ be open, $P$ be an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ a $\mathcal{I}^2$ function and suppose that the following conditions hold:*

*(1) For each $(x,y) \in D$,*

$$f(x) = \sum_{i=1}^{n} h_i(x,y) f_i(g_i(x,y));$$

*(2) $h_i$ is continuously differentiable for $i = 1, \dots, n$;*

*(3) the function $f_i$ is in $\mathcal{F}_{k+l}(X_i, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$, $(i = 1, 2, \dots, n)$;*

*(4) $g_i$ is $\mathcal{C}^2$ on $D$ $(i = 1, 2, \dots, n)$;*

*(5) for each $u_0 \in U$ there exists a $y_0$ such that $(\varphi(u_0, p_0), y_0) \in D$ and the rank of the derivative of*

$$(u,y) \mapsto g_i(\varphi(u,p_0), y)$$

*at $(u_0, y_0)$ is $k + l$ for each $1 \le i \le n$.*

*Then for any function $w : U \times P \to X$ that belongs to $\mathcal{K}^1$ the parametric integral*

$$p \mapsto \int_U w(u,p) f(\varphi(u,p))\, du$$

*is continuously differentiable on a neighbourhood of $p_0$.*

**Proof.** Let us choose open neighbourhoods $U_0$, $P_0$, $Y_0$ of $u_0$, $p_0$, $y_0$ such that $(\varphi(u,p),y)$ is in $D$ whenever $u\in U_0$, $p\in P_0$, $y\in Y_0$. We can also ensure that the rank of the derivative of the mapping $(u,y)\mapsto g_i(\varphi(u,p),y)$ is equal to $k+l$ for all $u\in U_0$, $p\in P_0$, $y\in Y_0$ and for $1\le i\le n$. This is possible, because $D$ is open, $g_i$ and $\varphi$ are $\mathcal{C}^2$ functions, the rank is lower semicontinuous and $U\times Y$ has dimension $k+l$, hence the rank cannot increase above $k+l$.

Now let $w: U_0\times P_0\to\mathbb{R}$ be a $\mathcal{K}^1$ function. Let us choose a $\mathcal{K}^1$ function $w_0: Y_0\to\mathbb{R}$ for which $\int_{Y_0} w_0(y)\,dy\neq 0$. By (1) we obtain

$$w(u,p)f(\varphi(u,p))w_0(y)=\sum_{i=1}^{n} w(u,p)w_0(y)h_i(\varphi(u,p),y)f_i(g_i(\varphi(u,p),y)).$$

Integrating both sides over $U_0\times Y_0$ we obtain

$$\begin{aligned}\int_{Y_0} w_0(y)\,dy\int_{U_0} w(u,p)f(\varphi(u,p))\,du\\ =\sum_{i=1}^{n}\int_{Y_0}\int_{U_0} w(u,p)w_0(y)h_i(\varphi(u,p),y)f_i(g_i(\varphi(u,p),y))\,du\,dy.\end{aligned}$$

Now the right hand side is in $\mathcal{C}^1$. This proves that

$$p\mapsto\int_{U_0} w(u,p)f(\varphi(u,p))\,du$$

is continuously differentiable on $P_0$. For an arbitrary $\mathcal{K}^1$ function $w: U\times P\to\mathbb{R}$ the statement follows using a partition of unity subordinate to an appropriate finite covering of the support of $u\mapsto w(u,p_0)$.

**3.2. Example.** Let us consider the functional equation

$$\sum_{i=0}^{n} h_i(x,y)f(x+g_i(y))=0,$$

whenever $x\in\mathbb{R}^m$, $y\in\mathbb{R}$. Suppose that the functions $h_i:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are continuously differentiable and the functions $g_i:\mathbb{R}\to\mathbb{R}^m$ are in $\mathcal{C}^2$. Introducing the variable $x_j=x+g_j(y)$ instead of $x$, we obtain

$$f(x_j)=-\sum_{i\neq j}\frac{h_i(x_j-g_j(y),y)}{h_j(x_j-g_j(y),y)}f(x_j-g_j(y)+g_i(y)). \tag{1}$$

To see that condition (5) from the previous theorem is satisfied we have to check the rank of the matrix

$$\begin{pmatrix}\dfrac{\partial\varphi_{p_0}^{(1)}}{\partial u_1}(u) & \dots & \dfrac{\partial\varphi_{p_0}^{(1)}}{\partial u_k}(u) & \dfrac{d\,g_i^{(1)}}{d\,y}(y)-\dfrac{d\,g_j^{(1)}}{d\,y}(y)\\ \vdots & & \vdots & \vdots\\ \dfrac{\partial\varphi_{p_0}^{(m)}}{\partial u_1}(u) & \dots & \dfrac{\partial\varphi_{p_0}^{(m)}}{\partial u_k}(u) & \dfrac{d\,g_i^{(m)}}{d\,y}(y)-\dfrac{d\,g_i^{(m)}}{d\,y}(y)\end{pmatrix}.$$

If this is $k+1$, then we may apply our theorem with $l = 1$. This means, geometrically, that the vector $g_i'(y) - g_j'(y)$ is not contained in the range of the linear operator $\varphi_{p_0}'(u)$ (which is known to be $k$-dimensional). This range can be any $k$-dimensional linear subspace in $\mathbb{R}^m$. It may happen that for each $k$-dimensional linear subspace, there exists a $y \in \mathbb{R}$ such that none of the vectors $g_i'(y) - g_j'(y)$, $i \neq j$ is contained in the linear subspace. Then our theorem can be applied directly and proves that $f \in \mathcal{F}_{k+1}(\mathbb{R}^m, \mathbb{R}, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ implies $f \in \mathcal{F}_k(\mathbb{R}^m, \mathbb{R}, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$. If this is the case for $k = m-1, m-2, \dots, 0$ then we obtain that every continuous solution is continuously differentiable. However, there are situations when this is not the case. If, for example, the derivative of the functions $g_i$ is constant, i. e. if $g_i(y) = ya_i + b_i$, then for any fixed $j$, equation (1) cannot be applied to get $f \in \mathcal{F}_k(\mathbb{R}^m, \mathbb{R}, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ from $f \in \mathcal{F}_{k+1}(\mathbb{R}^m, \mathbb{R}, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$, because for some $\varphi$'s the range of $\varphi_{p_0}'(u)$ will contain some of the vectors $g_i'(y) - g_j'(y) = a_i - a_j$. But we have the possibility to use *any* of the equations (1). Using the locality principle mentioned in the definition, it is enough to prove that for any $k$-dimensional linear subspace of $\mathbb{R}^n$ there exists a $j$ such that none of the vectors $a_i - a_j$, $i \neq j$ is contained in the given subspace. For example this is the situation if $n \geq m$ and the vectors $a_0, \dots, a_n$ are in general position. If this condition is not satisfied, then it is still possible that our theorem can be applied. A similar (but somewhat simpler) situation was studied in the paper [15], in the proof of Theorem 2.3.

**3.3. Remark.** Although, as the example above shows, Theorem 3.1 can be applied in several cases, it is not satisfying because condition (5) is too strong. If we want to apply Theorem 3.1 to prove that $f \in \mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$ then we have to allow an arbitrary $\varphi$. Hence condition (5) implicitly means that the rank of $\dfrac{\partial g_i}{\partial x}$ has to be large, even if $\dfrac{\partial g_i}{\partial y}$ has a large rank. This in practice means that the $g_i$ have to depend on all coordinates of $x$, which is not comfortable. We want to relax this condition. Instead of supposing that

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

has maximal possible rank $k + l$ at $(u_0, y_0)$, we only suppose that it has rank not less then some $k_i$.

**3.4. Lemma.** *Let $X$ be an open subset of $\mathbb{R}^n$, let $0 \leq k \leq n$ and let*

$$f \in \mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1).$$

*Let $U$ and $P$ be open subsets of Euclidean spaces and suppose that for the $\mathcal{C}^2$ mapping $\varphi : U \times P \to X$ the derivative of the partial mapping $u \mapsto \varphi(u, p)$ has rank not less than $k$ at the point $(u_0, p_0) \in U \times P$. Then there exists a neighbourhood $U_0$ of $u_0$ and $P_0$ of $p_0$ such that for any function $w : U \times P \to \mathbb{R}$ having support contained in $U_0 \times P_0$ the parametric integral*

$$p \mapsto \int_U w(u, p) f(\varphi(u, p))\, du$$

*is continuously differentiable on $P$.*

**Proof.** Denoting by $u_1$ the first $k$ coordinates of $u$, and by $u_2$ the rest of the coordinates of $u$, we may assume that the mapping $u_1 \mapsto \varphi(u_1, u_2, p)$ has nonzero Jacobian at $(u_0, p_0)$. Let us choose a neighbourhood $U_0 = U_1 \times U_2$ of $u_0$ and $P_0$ of $p_0$ such that this Jacobian is non-zero on $U_0 \times P_0$. Then the mapping

$$(u_2, p) \mapsto \int_{U_1} w(u_1, u_2, p) f(\varphi(u_1, u_2, p))\, du_1$$

is in $\mathcal{C}^1$. Integrating with respect to $u_2$, we obtain the statement of the lemma.

**3.5. Corollary.** *If $0 \leq k \leq l \leq n$ then*

$$\mathcal{F}_k(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1) \subset \mathcal{F}_l(X, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1).$$

The next theorem is our generalization of theorem 3.1.

**3.6. Theorem.** *Let $X$, $Y$ and $X_i$, $1 \leq i \leq n$ be open subsets of Euclidean spaces. Let $D$ be an open subset of $X \times Y$. Consider the functions $f : X \to \mathbb{R}^m$, $f_i : X_i \to \mathbb{R}^m$, $h_i : D \to \mathbb{R}$, $g_i : D \to X_i$ $(i = 1, 2, \ldots, n)$. Let $U \subset \mathbb{R}^k$ be open, $P$ be an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ an $\mathcal{I}^2$ function and suppose that the following conditions hold:*

*(1) For each $(x, y) \in D$,*

$$f(x) = \sum_{i=1}^{n} h_i(x, y) f_i(g_i(x, y));$$

*(2) $h_i$ is continuously differentiable for $i = 1, \ldots, n$;*
*(3) the function $f_i$ is in $\mathcal{F}_{k_i}(X_i, \mathbb{R}^m, \mathcal{K}^1, \mathcal{E}^2; \mathcal{C}^1)$, $(i = 1, 2, \ldots, n)$;*
*(4) $g_i$ is $\mathcal{C}^2$ on $D$ $(i = 1, 2, \ldots, n)$;*
*(5) for each $u_0 \in U$ there exists a $y_0$ such that $(\varphi(u_0, p_0), y_0) \in D$ and the rank of the derivative of*

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

*at $(u_0, y_0)$ is at least $k_i$ for each $1 \leq i \leq n$.*

*Then for any function $w : U \times P \to X$ that belongs to $\mathcal{K}^1$, the parametric integral*

$$p \mapsto \int_U w(u, p) f(\varphi(u, p))\, du$$

*is continuously differentiable on a neighbourhood of $p_0$.*

**Proof.** Let us choose open neighbourhoods $U_0$, $P_0$, $Y_0$ of $u_0$, $p_0$, $y_0$, respectively, such that $(\varphi(u,p),y)$ is in $D$ whenever $u \in U_0$, $p \in P_0$, $y \in Y_0$. Moreover, we can ensure that the rank of the derivative of the mapping $(u,y) \mapsto g_i(\varphi(u,p),y)$ is not less than $k_i$ for all $u \in U_0$, $p \in P_0$, $y \in Y_0$ and for $1 \le i \le n$. This is possible, because $D$ is open, $g_i$ and $\varphi$ are $\mathcal{C}^2$ functions, and the rank is lower semicontinuous.

Now let $w : U_0 \times P_0 \to \mathbb{R}$ be a $\mathcal{K}^1$ function. Let us choose a $\mathcal{K}^1$ function $w_0 : Y_0 \to \mathbb{R}$, for which $\int_{Y_0} w_0(y)\,dy \neq 0$. From (1) we obtain

$$w(u,p)f(\varphi(u,p))w_0(y) = \sum_{i=1}^{n} w(u,p)w_0(y)h_i(\varphi(u,p),y)f_i(g_i(\varphi(u,p),y)).$$

Integrating both sides over $U_0 \times Y_0$ we obtain that

$$\begin{aligned}&\int_{Y_0} w_0(y)\,dy \int_{U_0} w(u,p)f(\varphi(u,p))\,du\\ &\quad= \sum_{i=1}^{n} \int_{Y_0}\int_{U_0} w(u,p)w_0(y)h_i(\varphi(u,p),y)f_i(g_i(\varphi(u,p),y))\,du\,dy.\end{aligned}$$

Now the right hand side is in $\mathcal{C}^1$ by the previous lemma. This proves that

$$p \mapsto \int_{U_0} w(u,p)f(\varphi(u,p))\,du$$

is continuously differentiable on $P_0$. For an arbitrary $\mathcal{K}^1$ function $w : U \times P \to \mathbb{R}$ the statement follows by using a partition of unity subordinate to an appropriate finite covering of the support of $u \mapsto w(u,p_0)$.

Our last theorem is about higher order derivates. Here the functional equation is allowed to be nonlinear.

**3.7. Theorem.** *Let $X$ be an open subset of $\mathbb{R}^s$. Let $Y$, $Z$ and $X_i$, $Z_i$, $1 \le i \le n$ be open subsets of Euclidean spaces. Let $D$ be an open subset of $X \times Y$. Consider the functions $f : X \to \mathbb{R}^m$, $f_i : X_i \to Z_i$, $h : D \times Z_1 \times \cdots Z_i \to Z$, $g_i : D \to X_i$ $(i = 1,2,\dots,n)$. Let $r \ge 1$ an integer. Let $U \subset \mathbb{R}^k$ be open, $P$ be an open subset of some Euclidean space, $p_0 \in P$, $\varphi : U \times P \to X$ an $\mathcal{I}^2$ function and suppose that the following conditions hold:*

*(1) For each $(x,y) \in D$,*

$$f(x) = h(x,y,f_1(g_1(x,y))\dots f_n(g_n(x,y)));$$

*(2) all partial derivates $\partial_x^{\alpha_0}\partial_{z_1}^{\alpha_1}\dots\partial_{z_n}^{\alpha_n}h$ are continuously differentiable whenever $0 \le |\alpha| \le r$, where $\alpha = (\alpha_0,\alpha_1,\dots,\alpha_n) \in \mathbb{N}^{n+1}$ and $|\alpha| = \sum_{i=0}^{n}\alpha_i$;*

*(3) all partial derivates of order $r$ of the function $f_i$ are in $\mathcal{F}_{k_i}(X_i,\mathbb{R}^m,\mathcal{K}^1,\mathcal{E}^2;\mathcal{C}^1)$, $(i = 1,2,\dots,n)$;*

*(4)* $g_i$ *is* $\mathcal{C}^{r+1}$ *on* $D$ $(i = 1, 2, \dots, n)$*;*
*(5)* *for each* $u_0 \in U$ *there exists a* $y_0$ *such that* $(\varphi(u_0, p_0), y_0) \in D$ *and the rank of the derivative of*

$$(u, y) \mapsto g_i(\varphi(u, p_0), y)$$

*at* $(u_0, y_0)$ *is* $k_i$ *for each* $1 \le i \le n$*.*

*Then for any function* $w : U \times P \to X$ *that belongs to* $\mathcal{K}^1$ *and for any multiindex* $\alpha \in \mathbb{N}^s$ *for which* $|\alpha| = r$*, the parametric integral*

$$p \mapsto \int_U w(u, p)(\partial^\alpha f)(\varphi(u, p))\, du$$

*is continuously differentiable on a neighbourhood of* $p_0$*.*

**Proof.** Let $1 \le q \le s$, and differentiate equation (1) partially with respect to $x_q$. We have, omitting the variables,

$$\frac{\partial f}{\partial x_q} = \frac{\partial h}{\partial x_q} + \sum_{i=1}^{n} \sum_{j} \frac{\partial h}{\partial z_{i,j}} \sum_{k=1}^{r_i} \frac{\partial f_{i,j}}{\partial x_{i,k}} \frac{\partial g_{i,k}}{\partial x_q}.$$

Here $z_i = (z_{i,j})$, $x_i = (x_{i,k})$, $f_i = (f_{i,j})$ and $g_i = (g_{i,k})$. The last equation shows, that whenever $\alpha \in \mathbb{N}^s$, $|\alpha| = 1$, the function $\partial^\alpha f$ satisfies the functional equation

$$\partial^\alpha f(x) = h_{\alpha,0}(x, y) + \sum_{\beta=1}^{n_\alpha} h_{\alpha,\beta}(x, y) f_{\alpha,\beta}\left(g_{\alpha,\beta}(x, y)\right)$$

for $(x, y) \in D$. Here, if the $q$-th coordinate of $\alpha$ equals one, and all other coordinates are zero, then

$$h_{\alpha,0}(x, y) = \frac{\partial h}{\partial x_q}\left(x, y, f_0(y), f_1\left(g_1(x, y)\right), \dots, f_n\left(g_n(x, y)\right)\right),$$

$$g_{\alpha,\beta} = g_i \qquad \text{for some } 1 \le i \le n,$$

$$f_{\alpha,\beta} = \frac{\partial f_{i,j}}{\partial x_{i,k}} \qquad \text{for some } i, j, k,$$

and

$$h_{\alpha,\beta}(x, y) = \frac{\partial h}{\partial z_{i,j}}\left(x, y, f_0(y), f_1\left(g_1(x, y)\right), \dots, f_n\left(g_n(x, y)\right)\right) \frac{\partial g_{i,k}}{\partial x_q}(x, y)$$

for some $i, j, k$.

It is clear that $h_{\alpha,\beta}$ has a continuous $p$-th partial derivative with respect to $x$ whenever $0 \le \beta \le n_\alpha$, moreover $f_{\alpha,\beta}$ maps some $X_i$ into $\mathbb{R}$ and all $p-1$-st partial derivatives of $f_{\alpha,\beta}$ exist. Repeating this process, we have by induction on $|\alpha|$, that if $\alpha \in \mathbb{N}^s$, $1 \le |\alpha| \le p$, then

$$\partial^\alpha f(x) = h_{\alpha,0}(x,y) + \sum_{\beta=1}^{n_\alpha} h_{\alpha,\beta}(x,y) f_{\alpha,\beta}\left(g_{\alpha,\beta}(x,y)\right) \tag{5}$$

whenever $(x,y) \in D$. Here $h_{\alpha,\beta} : D \to Z$ is continuous, its $p+1-|\alpha|$-th partial derivative with respect to $x$ is continuous moreover $f_{\alpha,\beta} : X_i \to \mathbb{R}$ for some $1 \le i \le n$ and all of its $p-|\alpha|$-th partial derivatives are continuous. Finally, $g_{\alpha,\beta} = g_i$ for the same $i$ for which $\operatorname{dmn} f_{\alpha,\beta} = X_i$.

Now we use Theorem 3.6. Then we have that for any function $w : U \times P \to X$ that belongs to $\mathcal{K}^1$, and for any multiindex $\alpha \in \mathbb{N}^s$, the parametric integral

$$p \mapsto \int_U w(u,p)(\partial^\alpha f)(\varphi(u,p))\, du$$

is continuously differentiable on a neighbourhood of $p_0$.

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