Kalkulus II.

Járai Antal

Ezek a programok csak szemléltetésre szolgálnak

1. Lineáris algebra

2. Többváltozós függvények

3. Függvénysorozatok és függvénysorok

3.1. Pontonkénti és egyenletes konvergencia

```
> restart;
```

3.1.1. Függvényterek, függvénysorozatok.

3.1.2. Cauchy-kritérium függvénysorozatokra.

3.1.3. Függvénysorok.

3.1.4. Weierstrass-kritérium.

3.1.5. Határátmenetek felcserélése.

```
> limit(limit(n/(n+m),n=infinity),m=infinity);
  limit(limit(n/(n+m),m=infinity),n=infinity);
```

$$1$$

$$0 \qquad (3.1.5.1)$$

3.1.6. Tétel.

3.1.7. Példa.

3.1.8. Példa.

**3.1.9. Monoton konvergencia tétel.*

**3.1.10. Dominált konvergencia tétel.*

3.1.11. Következmény: határátmenet és integrál felcserélése.

3.1.12. Következmény.

3.1.13. Példa.

```
> limit(n*x*(1-x^2)^n,n=infinity) assuming(0<x, x<1);
```

$$0 \qquad (3.1.13.1)$$

```
> int(n*x*(1-x^2)^n,x=0..1) assuming(n::posint);
```

$$\frac{1}{2}\,\frac{n}{n+1} \qquad (3.1.13.2)$$

```
> limit(n^2*x*(1-x^2)^n,n=infinity) assuming(0<x, x<1);
```

$$0 \qquad (3.1.13.3)$$

```
> int(n^2*x*(1-x^2)^n,x=0..1) assuming(n::posint);
```

$$\frac{1}{2}\,\frac{n^2}{n+1} \qquad (3.1.13.4)$$

3.1.14. Bevezető példa.

```
> f:=sin(n*x)/n^(1/2); limit(f,n=infinity);
```

$$f := \frac{\sin(n\,x)}{\sqrt{n}}$$

$$0 \qquad (3.1.14.1)$$

```
> diff(f,x); limit(%,n=infinity);
```

$$\cos(n\,x)\,\sqrt{n}$$

$$\mathit{undefined} \qquad (3.1.14.2)$$

3.1.15. Függvénysorozat tagonkénti differenciálása.

*3.1.16. Tétel.

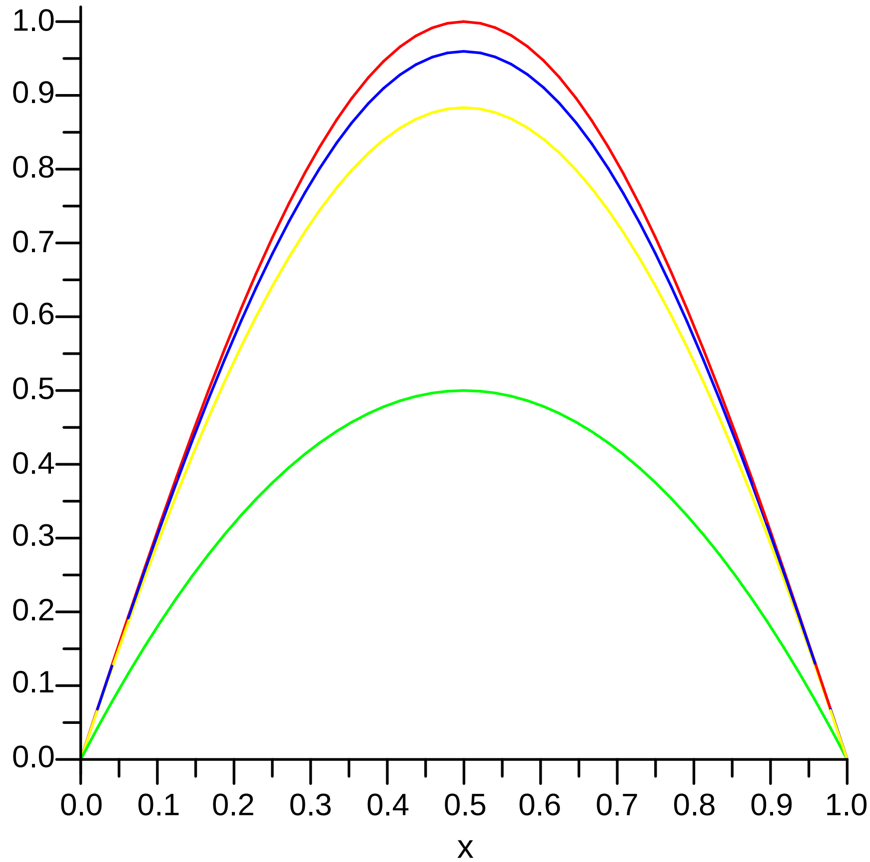
*3.1.17. Weierstrass approximációs tétele.

```
> f:=x->sin(Pi*x);
  B:=(n,x)->sum(binomial(n,k)*f(k/n)*x^k*(1-x)^(n-k),k=0..n);
```

$$f := x \rightarrow \sin(\pi\,x)$$

$$B := (n, x) \rightarrow \sum_{k=0}^{n} \mathrm{binomial}(n, k)\, f\left(\frac{k}{n}\right) x^k\,(1-x)^{n-k} \qquad (3.1.17.1)$$

```
> plot({f(x),B(2,x),B(10,x),B(30,x)},x=0..1);
```



3.2. Hatványsorok

```
> restart;
```

3.2.1. Definíció.
3.2.2. Cauchy-Hadamard-tétel.
3.2.3. Definíció.
3.2.4. Következmény.
3.2.5. Tétel.
3.2.6. Taylor-tétel.
3.2.7. Következmény.

```
> sum(x^n,n=0..infinity); sum(x^n/n,n=1..infinity);
  sum(x^n/n^2,n=1..infinity);
```

```
series(%%%,x=0); series(%%%,x=0); series(%%%,x=0);
```

$$-\frac{1}{x-1}$$

$$-\ln(1-x)$$

$$\mathrm{polylog}(2, x)$$

$$1+x+x^2+x^3+x^4+x^5+\mathrm{O}(x^6)$$

$$x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\mathrm{O}(x^6)$$

$$x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+\mathrm{O}(x^6) \tag{3.2.7.1}$$

3.2.8. Megjegyzés.

3.2.9. Megjegyzés.

```
> series(1/(1-x),x=1/2);
```

$$2+4\left(x-\frac{1}{2}\right)+8\left(x-\frac{1}{2}\right)^2+16\left(x-\frac{1}{2}\right)^3+32\left(x-\frac{1}{2}\right)^4+64\left(x-\frac{1}{2}\right)^5+\mathrm{O}\left(\left(x-\frac{1}{2}\right)^6\right) \tag{3.2.9.1}$$

3.2.10. Az exponenciális függvény, trigonometrikus és hiperbolikus függvények.

```
> sum(z^n/n!,n=0..infinity); exp(z); exp(z1+z2); expand(%);
```

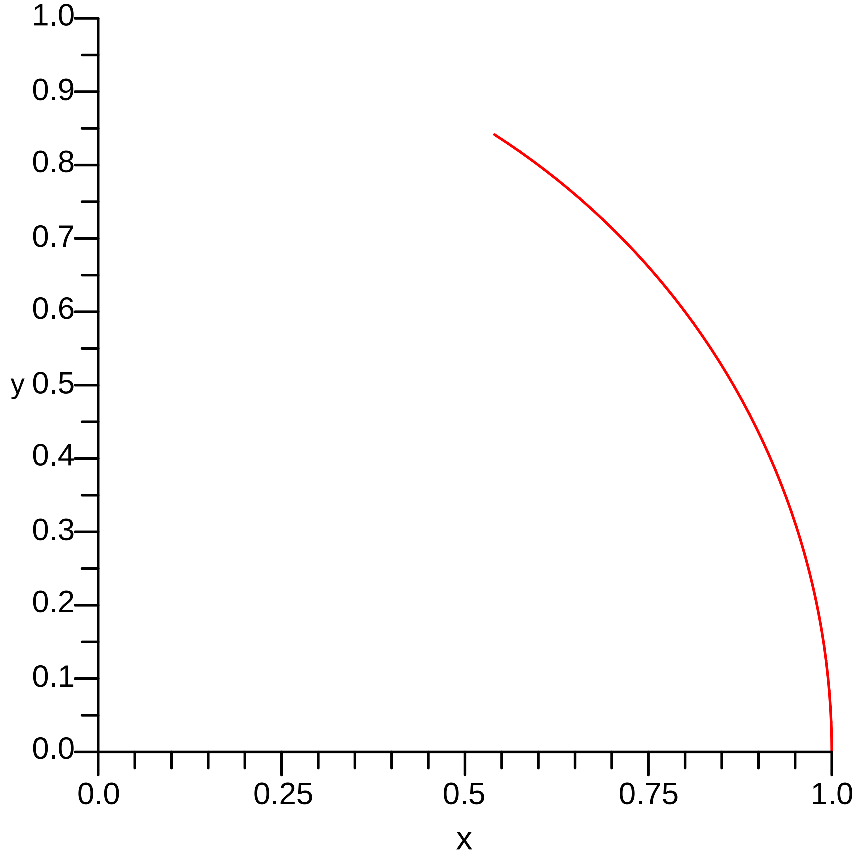
$$\mathrm{e}^z$$

$$\mathrm{e}^z$$

$$\mathrm{e}^{z1+z2}$$

$$\mathrm{e}^{z1}\,\mathrm{e}^{z2} \tag{3.2.10.1}$$

```
> plots[complexplot](exp(I*t),t=0..1,x=0..1,y=0..1);
```




```
> (exp(I*z)+exp(-I*z))/2; convert(%,trig);
  (exp(I*z)-exp(-I*z))/(2*I); convert(%,trig);
```

$$\frac{1}{2}\,\mathrm{e}^{\mathrm{I}\,z} + \frac{1}{2}\,\mathrm{e}^{-\mathrm{I}\,z}$$

$$\cos(z)$$

$$-\frac{1}{2}\,\mathrm{I}\left(\mathrm{e}^{\mathrm{I}\,z} - \mathrm{e}^{-\mathrm{I}\,z}\right)$$

$$\sin(z) \tag{3.2.10.2}$$

```
> sum((-1)^n*z^(2*n)/(2*n)!,n=0..infinity);
  sum((-1)^n*z^(2*n+1)/(2*n+1)!,n=0..infinity);
```

$$\cos(z)$$

$$\sin(z) \tag{3.2.10.3}$$

```
> exp(I*t); convert(%,trig);
```

$$\mathrm{e}^{\mathrm{I}\,t}$$

$$\cos(t) + \mathrm{I}\sin(t) \tag{3.2.10.4}$$

```
> cos(z)^2+sin(z)^2; simplify(%);
```
$$\cos(z)^2+\sin(z)^2$$
$$1 \tag{3.2.10.5}$$

```
> cos(z1+z2); expand(%); sin(z1+z2); expand(%);
```
$$\cos(z1+z2)$$
$$\cos(z1)\cos(z2)-\sin(z1)\sin(z2)$$
$$\sin(z1+z2)$$
$$\sin(z1)\cos(z2)+\cos(z1)\sin(z2) \tag{3.2.10.6}$$

```
> (exp(z)+exp(-z))/2; convert(%,trigh);
  (exp(z)-exp(-z))/2; convert(%,trigh);
```
$$\frac{1}{2}\,e^{z}+\frac{1}{2}\,e^{-z}$$
$$\cosh(z)$$
$$\frac{1}{2}\,e^{z}-\frac{1}{2}\,e^{-z}$$
$$\sinh(z) \tag{3.2.10.7}$$

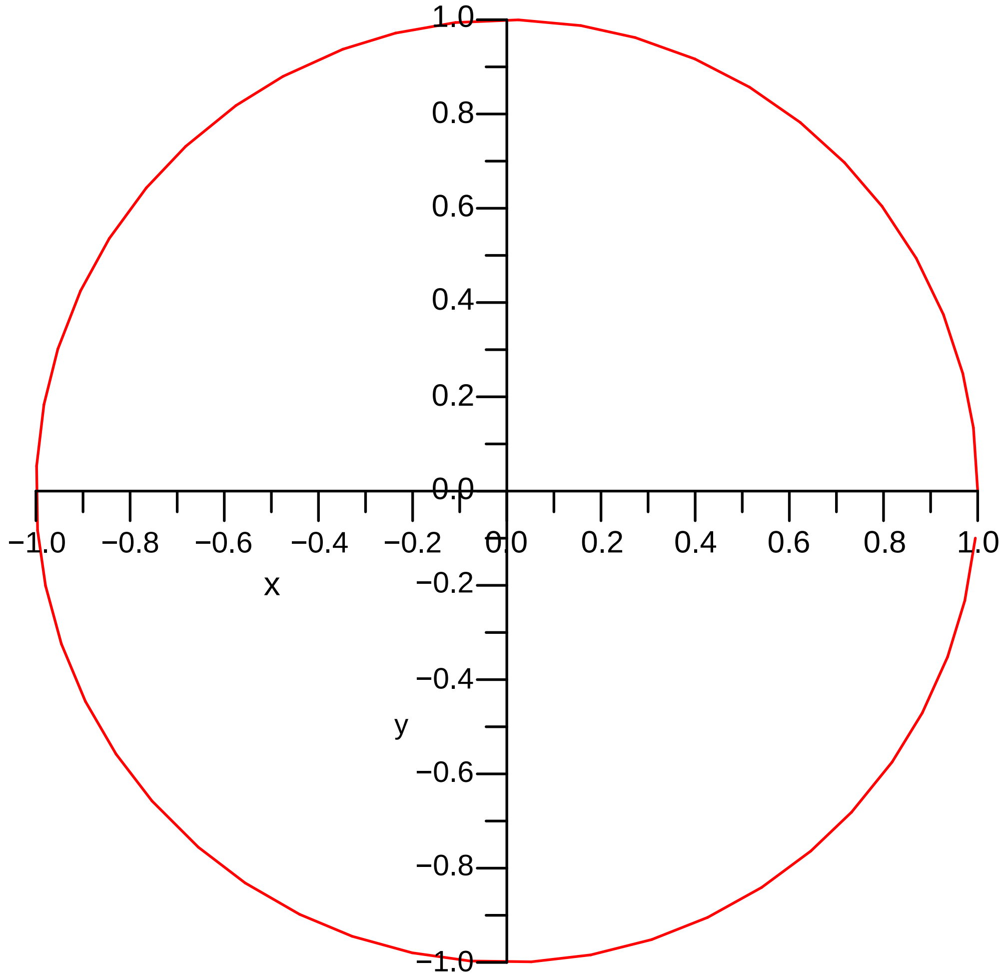
```
> sum(z^(2*n)/(2*n)!,n=0..infinity);
  sum(z^(2*n+1)/(2*n+1)!,n=0..infinity);
```
$$\cosh(z)$$
$$\sinh(z) \tag{3.2.10.8}$$

```
> cosh(z)^2-sinh(z)^2; simplify(%);
```
$$\cosh(z)^2-\sinh(z)^2$$
$$1 \tag{3.2.10.9}$$

```
> cosh(z1+z2); expand(%); sinh(z1+z2); expand(%);
```
$$\cosh(z1+z2)$$
$$\cosh(z1)\cosh(z2)+\sinh(z1)\sinh(z2)$$
$$\sinh(z1+z2)$$
$$\sinh(z1)\cosh(z2)+\cosh(z1)\sinh(z2) \tag{3.2.10.10}$$

3.3.11. Tétel.

```
> plots[complexplot](exp(I*t),t=0..2*Pi-0.1,x=-1..1,y=-1..1);
```




```
> exp(Pi*I/2); exp(Pi*I); cos(z+2*Pi); sin(z+2*Pi);
```

$$I$$

$$-1$$

$$\cos(z)$$

$$\sin(z) \tag{3.2.11.1}$$

```
> exp(z+2*Pi*I); expand(%);
```

$$e^{z+2I\pi}$$

$$e^{z} \tag{3.2.11.2}$$

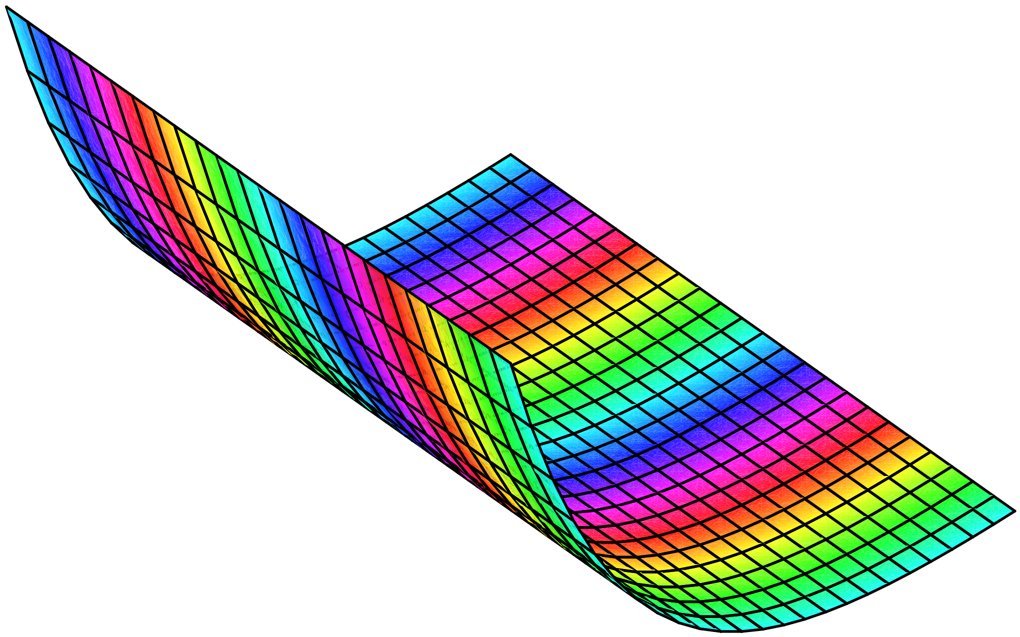
```
> cosh(z+2*Pi*I); sinh(z+2*Pi*I);
```

$$\cosh(z)$$

$$\sinh(z) \tag{3.2.11.3}$$

3.3.12. Feladat.

```
> plots[complexplot3d](exp(z),z=-2-2*Pi*I..2+2*Pi*I);
```



3.2.13. Definíció.

```
> ln(x+I*y); evalc(%);
```

$$\ln(x + \mathrm{I}\,y)$$

$$\frac{1}{2}\ln(x^2 + y^2) + \mathrm{I}\arctan(y, x) \tag{3.2.13.1}$$

3.2.14. Tétel.

```
> D(exp); D(sin); D(cos); D(sinh); D(cosh);
```

$$\exp$$

$$\cos$$

$$-\sin$$

$$\cosh$$

$$\sinh \quad (3.2.14.1)$$

```
> D(tan); D(cot); D(tanh); D(coth);
```

$$1+\tan^2$$

$$-1-\cot^2$$

$$1-\tanh^2$$

$$1-\coth^2 \quad (3.2.14.2)$$

```
> D(log); D(z->z^alpha);
```

$$z \to \frac{1}{z}$$

$$z \to \frac{z^{\alpha}\,\alpha}{z} \quad (3.2.14.3)$$

3.2.15. Mértani sor.

```
> sum(z^n,n=0..infinity);
```

$$-\frac{1}{z-1} \quad (3.2.15.1)$$

3.2.16. A logaritmus függvény hatványsorai.

```
> sum(z^n/n,n=1..infinity);
```

$$-\ln(1-z) \quad (3.2.16.1)$$

```
> sum((-1)^(n+1)*z^n/n,n=1..infinity);
```

$$\ln(1+z) \quad (3.2.16.2)$$

```
> sum(z^(2*n+1)/(2*n+1),n=0..infinity);
```

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \quad (3.2.16.3)$$

3.2.17. Az arctg függvény hatványsora.

```
> sum((-1)^(n+1)*z^(2*n+1)/(2*n+1),n=0..infinity);
```

$$\frac{1}{2}\,\mathrm{I}\ln\left(\frac{1+\mathrm{I}\,z}{1-\mathrm{I}\,z}\right) \quad (3.2.17.1)$$

3.2.18. Newton binomiális sora.

```
> sum(binomial(alpha,n)*z^n,n=0..infinity);
```

$$(1+z)^{\alpha} \quad (3.2.18.1)$$

3.2.19. *Az arcsin függvény hatványsora.*

```
> sum((-1)^n*binomial(-1/2,n)*z^(2*n+1)/(2*n+1),n=0..
infinity);
```

$$\frac{z\arcsin\left(\sqrt{z^2}\right)}{\sqrt{z^2}} \tag{3.2.19.1}$$

3.2.20. *Hatványsorok alkalmazása differenciálegyenletek megoldására.*

```
> dsolve(x^2*diff(y(x),x,x)+x*diff(y(x),x)+(x^2-n^2)*y(x)=0);
```

$$y(x) = \_C1\,\mathrm{BesselJ}(n, x) + \_C2\,\mathrm{BesselY}(n, x) \tag{3.2.20.1}$$

3.3. Fourier-sorok

```
> restart;
```

3.3.1. *Általánosított Pitagorasz-tétel.*

3.3.2. *Fourier-sor.*

3.3.3. *Tétel.*

3.3.4. *Tétel.*

3.3.5. *Definíció.*

3.3.6. *Definíció.*

3.3.7. *Riesz-Fischer-tétel.*

3.3.8. *Klasszikus Fourier-sorok.*

```
> f:=x->abs(x); l:=1/2; a0:=1/l*int(f(x),x=-l..l);
a:=k->(1/l*int(f(x)*cos(Pi*k*x/l),x=-l..l) assuming
(k::posint));
b:=k->(1/l*int(f(x)*sin(Pi*k*x/l),x=-l..l) assuming
(k::posint));
eval(a(k)); eval(b(k));
```

$$f := x \rightarrow |x|$$

$$l := \frac{1}{2}$$

$$a0 := \frac{1}{2}$$

$$a := k \rightarrow assuming\left(\left[\frac{\displaystyle\int_{-l}^{l} f(x)\cos\left(\frac{\pi\, k\, x}{l}\right) \mathrm{d}x}{l}\right], [k::posint]\right)$$

$$b := k \rightarrow assuming\left(\left[\frac{\displaystyle\int_{-l}^{l} f(x)\sin\left(\frac{\pi\, k\, x}{l}\right) \mathrm{d}x}{l}\right], [k::posint]\right)$$

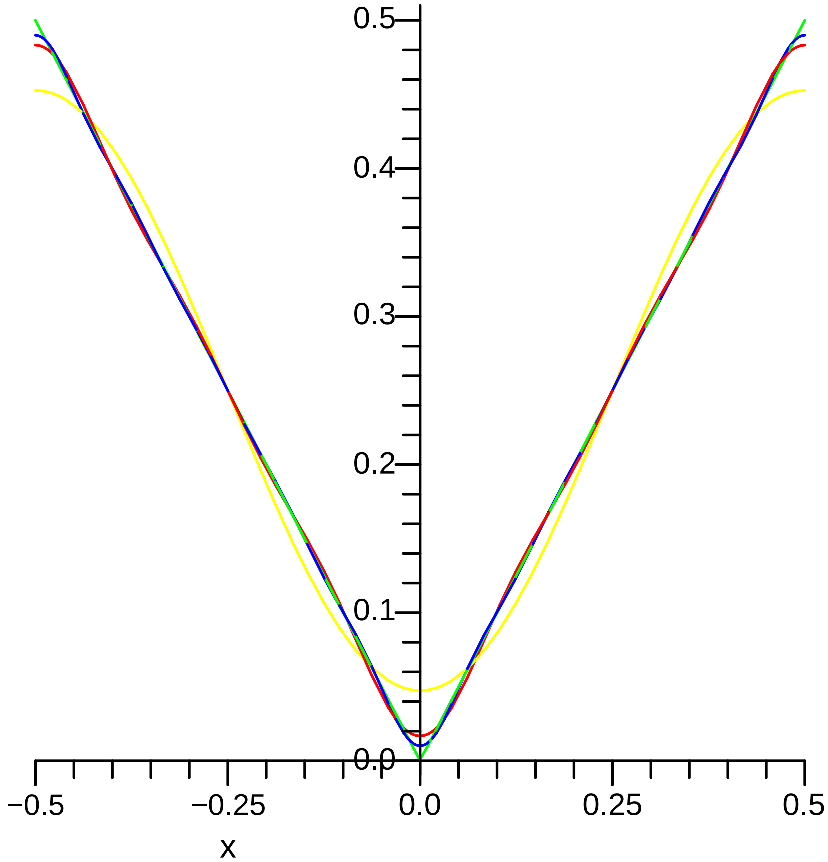
$$\frac{(-1)^k - 1}{\pi^2\, k^2}$$

$$0 \tag{3.3.8.1}$$

```
> s:=(n,x)->a0/2+sum(a(k)*cos(Pi*k*x/l)+b(k)*sin(Pi*k*x/l),k=
  1..n);
```

$$s := (n, x) \rightarrow \frac{1}{2}\, a0 + \sum_{k=1}^{n}\left(a(k)\cos\left(\frac{\pi\, k\, x}{l}\right) + b(k)\sin\left(\frac{\pi\, k\, x}{l}\right)\right) \tag{3.3.8.2}$$

```
> plot({f(x),s(2,x),s(5,x),s(10,x)},x=-l..l);
```




```
> f:=x->signum(x); l:=1/2; a0:=1/l*int(f(x),x=-l..l);
  a:=k->(1/l*int(f(x)*cos(Pi*k*x/l),x=-l..l) assuming
  (k::posint));
  b:=k->(1/l*int(f(x)*sin(Pi*k*x/l),x=-l..l) assuming
  (k::posint));
  eval(a(k)); eval(b(k));
```

$$f := x \to \mathrm{signum}(x)$$

$$l := \frac{1}{2}$$

$$a0 := 0$$

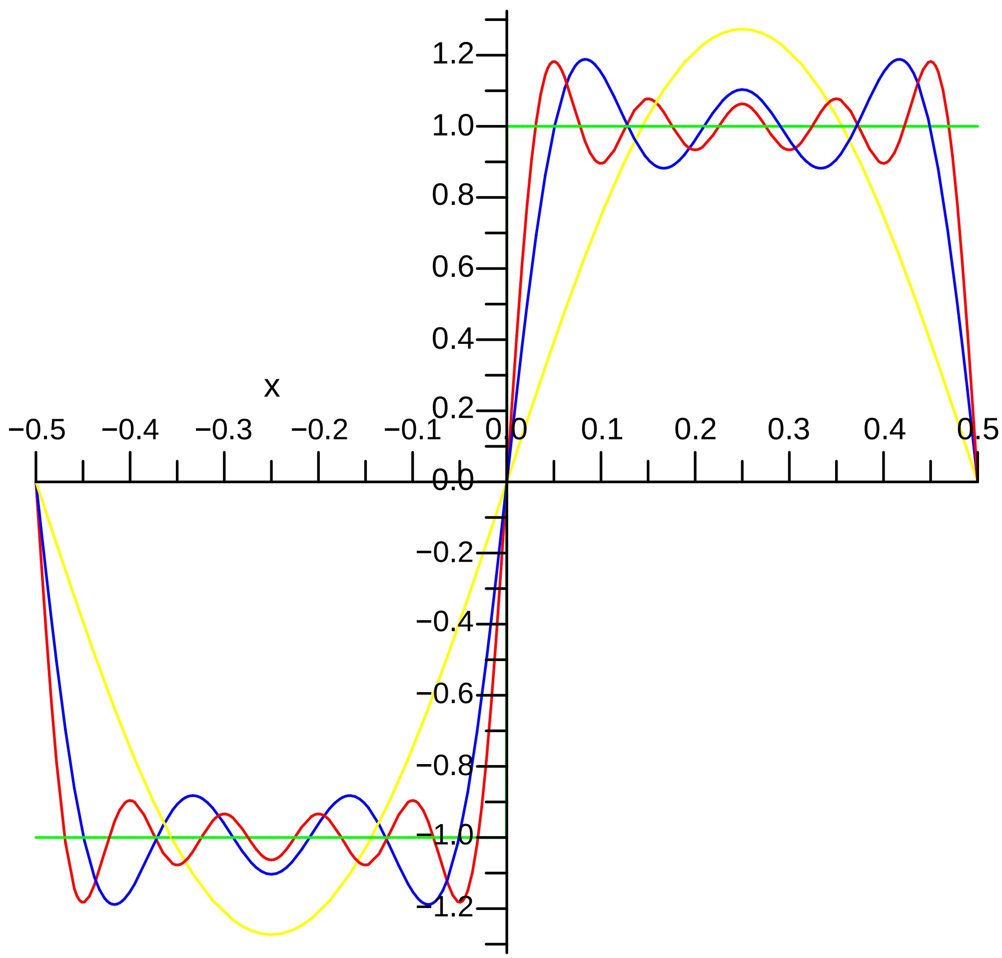
$$a := k \to assuming\left(\left[\frac{\int_{-l}^{l} f(x)\cos\left(\frac{\pi\, k\, x}{l}\right) \mathrm{d}x}{l}\right], [k{::}posint]\right)$$

$$b := k \to assuming\left(\left[\frac{\int_{-l}^{l} f(x)\sin\left(\frac{\pi k x}{l}\right)dx}{l}\right], [k{:}posint]\right)$$

$$0$$

$$-\frac{2\left(\left(-1\right)^{k}-1\right)}{\pi k} \tag{3.3.8.3}$$

```
> plot({f(x),s(2,x),s(5,x),s(10,x)},x=-l..l);
```



▶ ***3.3.9. Tétel.***

▶ ***3.3.10. Dirichlet-formula.***

▶ ***3.3.11. Lipschitz-kritérium.***

▶ ***3.3.12. Következmény.***

▶ ***3.3.13. Következmény: Riemann-féle lokalizációs tétel.***

3.3.14. *Klasszikus ortogonális polinomok.*

```
> with(orthopoly);
```

$$[G, H, L, P, T, U] \tag{3.3.14.1}$$

```
> JacobiP(0,1,3/4,x); p0:=simplify(%,'JacobiP');
  JacobiP(1,1,3/4,x); p1:=simplify(%,'JacobiP');
  JacobiP(2,1,3/4,x); p2:=simplify(%,'JacobiP');
  JacobiP(3,1,3/4,x); p3:=simplify(%,'JacobiP');
```

$$\mathrm{JacobiP}\left(0, 1, \frac{3}{4}, x\right)$$

$$p0 := 1$$

$$\mathrm{JacobiP}\left(1, 1, \frac{3}{4}, x\right)$$

$$p1 := \frac{1}{8} + \frac{15}{8}\, x$$

$$\mathrm{JacobiP}\left(2, 1, \frac{3}{4}, x\right)$$

$$p2 := -\frac{33}{8} + \frac{57}{8}\, x + \frac{437}{128}\, (x-1)^2$$

$$\mathrm{JacobiP}\left(3, 1, \frac{3}{4}, x\right)$$

$$p3 := -\frac{53}{4} + \frac{69}{4}\, x + \frac{621}{32}\, (x-1)^2 + \frac{6417}{1024}\, (x-1)^3 \tag{3.3.14.2}$$

```
> int((1-x)^1*(1+x)^(3/4)*p1*p2,x=-1..1);
```

$$0 \tag{3.3.14.3}$$

```
> P(0,x); P(1,x); P(2,x); P(3,x);
```

$$1$$

$$x$$

$$-\frac{1}{2} + \frac{3}{2}\, x^2$$

$$\frac{5}{2}\, x^3 - \frac{3}{2}\, x \tag{3.3.14.4}$$

```
> p0:=P(0,3/4,3/4,x); p1:=P(1,3/4,3/4,x);
  p2:=P(2,3/4,3/4,x); p3:=P(3,3/4,3/4,x);
```

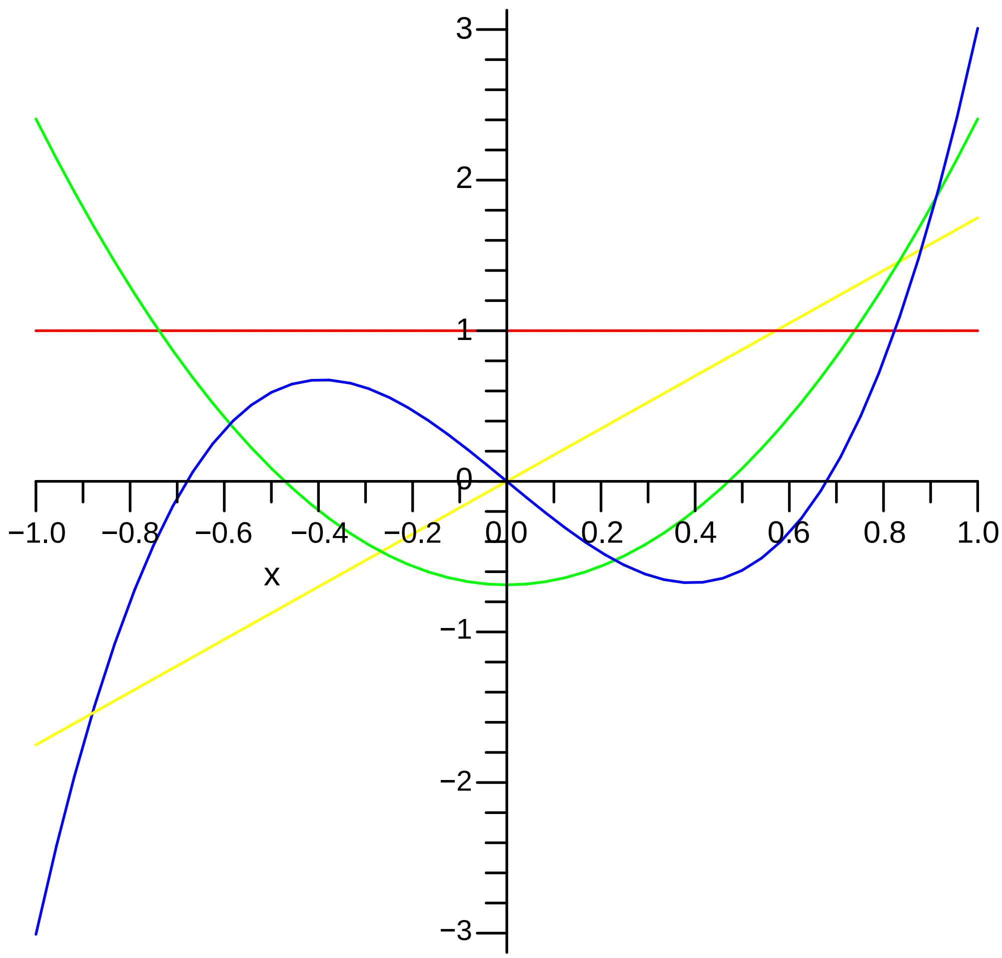
$$p0 := 1$$

$$p1 := \frac{7}{4}\, x$$

$$p2 := -\frac{121}{32} + \frac{99}{16}\,x + \frac{99}{32}\,(x-1)^2$$

$$p3 := -\frac{715}{64} + \frac{1815}{128}\,x + \frac{2145}{128}\,(x-1)^2 + \frac{715}{128}\,(x-1)^3 \qquad (3.3.14.5)$$

```
> plot({p0,p1,p2,p3},x=-1..1);
```




```
> LaguerreL(0,0,x); p0:=simplify(%,'LaguerreL');
  LaguerreL(1,0,x); p1:=simplify(%,'LaguerreL');
  LaguerreL(2,0,x); p2:=simplify(%,'LaguerreL');
  LaguerreL(3,0,x); p3:=simplify(%,'LaguerreL');
```

$$\mathrm{LaguerreL}(0, x)$$

$$p0 := 1$$

$$\mathrm{LaguerreL}(1, x)$$

$$p1 := 1 - x$$

$$\mathrm{LaguerreL}(2, x)$$

$$p2 := 1 - 2\,x + \frac{1}{2}\,x^2$$

$$\mathrm{LaguerreL}(3, x)$$

$$p3 := 1 - 3\,x + \frac{3}{2}\,x^2 - \frac{1}{6}\,x^3 \qquad (3.3.14.6)$$

```
> HermiteH(0,x); p0:=simplify(%,'HermiteH');
  HermiteH(1,x); p1:=simplify(%,'HermiteH');
  HermiteH(2,x); p2:=simplify(%,'HermiteH');
  HermiteH(3,x); p3:=simplify(%,'HermiteH');

  plot({p0,p1,p2,p3},x=0..infinity);
```

$$\mathrm{HermiteH}(0, x)$$

$$p0 := 1$$

$$\mathrm{HermiteH}(1, x)$$

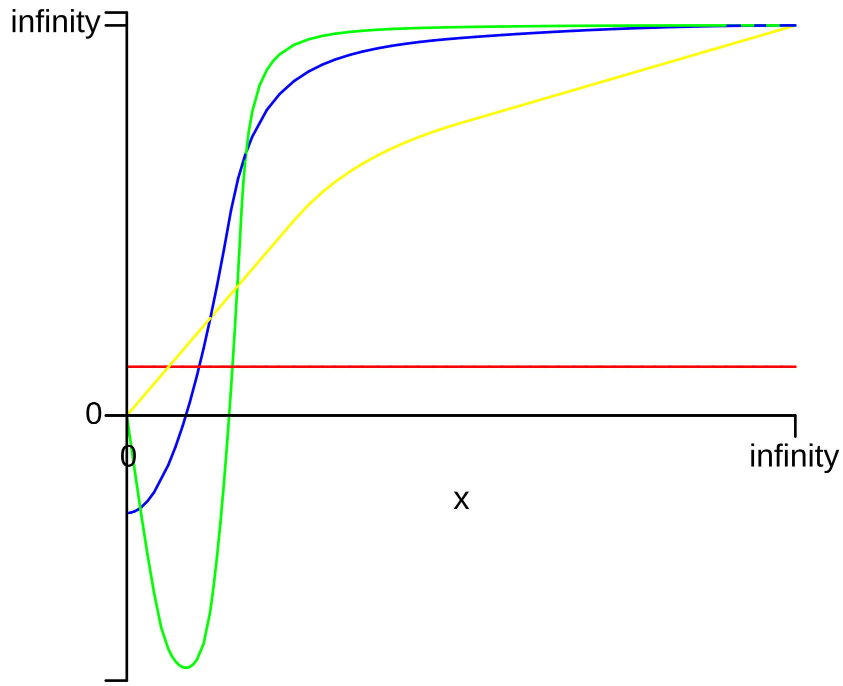
$$p1 := 2\,x$$

$$\mathrm{HermiteH}(2, x)$$

$$p2 := -2 + 4\,x^2$$

$$\mathrm{HermiteH}(3, x)$$

$$p3 := 8\,x^3 - 12\,x$$


infinity
0
0
infinity
x