Kalkulus II.

Járai Antal

Ezek a programok csak szemléltetésre szolgálnak

1. Lineáris algebra

```
> restart;
```

1.1. Mátrixok és vektorok

1.1.1. Test.

```
> 1,2+2,3^2,3.45/3,Pi,3+4*I,x^2=-1,Alma,infinity,-infinity;
```

$$1,\ 4,\ 9,\ 1.150000000,\ \pi,\ 3+4\,\mathrm{I},\ x^2=-1,\ \mathit{Alma},\ \infty,\ -\infty \tag{1.1.1.1}$$

```
> evalf(%);
```

$$1.,\ 4.,\ 9.,\ 1.150000000,\ 3.141592654,\ 3.+4.\,\mathrm{I},\ x^2=-1.,\ \mathit{Alma},\ \mathit{Float}(\infty),\ -\mathit{Float}(\infty) \tag{1.1.1.2}$$

```
> %%[5],%%[6];
```

$$\pi,\ 3+4\,\mathrm{I} \tag{1.1.1.3}$$

```
> ?Pi
> Digits:=50;
```

$$\mathit{Digits} := 50 \tag{1.1.1.4}$$

```
> evalf(Pi);
```

$$3.1415926535897932384626433832795028841971693993751 \tag{1.1.1.5}$$

```
> Digits:=10;
```

$$\mathit{Digits} := 10 \tag{1.1.1.6}$$

```
> A:=Alma; Alma:=1; A; Alma:='Alma'; A;
```

$$A := \mathit{Alma}$$

$$\mathit{Alma} := 1$$

$$1$$

$$\mathit{Alma} := \mathit{Alma}$$

$$\mathit{Alma} \tag{1.1.1.7}$$

```
> whattype(1); whattype(1/2); whattype(0.5); whattype(Alma);
  whattype(infinity);whattype(Pi);whattype(A);
  whattype(2); whattype(krikszkraksz); whattype
  ("krikszkraksz");
```

$$integer$$
$$fraction$$
$$float$$
$$symbol$$
$$extended\_numeric$$
$$symbol$$
$$symbol$$
$$integer$$
$$symbol$$
$$string \qquad (1.1.1.8)$$

```
> type(1,integer);type(1,float);type(1+3*I,complexcons);
```

$$true$$
$$false$$
$$true \qquad (1.1.1.9)$$

```
> z:=3+4*I;Re(z);Im(z);abs(z);
```

$$z := 3 + 4\,\mathrm{I}$$
$$3$$
$$4$$
$$5 \qquad (1.1.1.10)$$

```
> solve(x^2=-1);
```

$$\mathrm{I}, -\mathrm{I} \qquad (1.1.1.11)$$

```
> solve(x^5+x+1=0);
```

$$-\frac{1}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{3}, -\frac{1}{2} - \frac{1}{2}\,\mathrm{I}\sqrt{3}, \qquad (1.1.1.12)$$

$$-\frac{1}{6}\left(100 + 12\sqrt{69}\right)^{1/3} - \frac{2}{3\left(100 + 12\sqrt{69}\right)^{1/3}} + \frac{1}{3},$$

$$\frac{1}{12}\left(100 + 12\sqrt{69}\right)^{1/3} + \frac{1}{3\left(100 + 12\sqrt{69}\right)^{1/3}} + \frac{1}{3}$$

$$+ \frac{1}{2}\,\mathrm{I}\sqrt{3}\left(-\frac{1}{6}\left(100 + 12\sqrt{69}\right)^{1/3} + \frac{2}{3\left(100 + 12\sqrt{69}\right)^{1/3}}\right),$$

$$\frac{1}{12}\left(100+12\sqrt{69}\right)^{1/3}+\frac{1}{3\left(100+12\sqrt{69}\right)^{1/3}}$$

$$+\frac{1}{3}-\frac{1}{2}\,\mathrm{I}\sqrt{3}\left(-\frac{1}{6}\left(100+12\sqrt{69}\right)^{1/3}\right.$$

$$\left.+\frac{2}{3\left(100+12\sqrt{69}\right)^{1/3}}\right)$$

```
> solve(x^5+x^2+1=0);
```

$$\mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=1\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=2\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=3\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=4\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=5\right) \quad (1.1.1.13)$$

```
> solve(x^7+x^6+x^5+x^4+x^3+2*x^2+x+1);
```

$$-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3},\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=1\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=2\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=3\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=4\right),\ \mathrm{RootOf}\left(\_Z^5+\_Z^2+1,\ index=5\right) \quad (1.1.1.14)$$

```
> evalf(%);
```

$$-0.5000000000+0.8660254040\,\mathrm{I},\ -0.5000000000-0.8660254040\,\mathrm{I},\ 0.7515192324+0.7846159210\,\mathrm{I},\ -0.1545896767+0.8280741332\,\mathrm{I},\ -1.193859111,\ -0.1545896767-0.8280741332\,\mathrm{I},\ 0.7515192324-0.7846159210\,\mathrm{I} \quad (1.1.1.15)$$

1.1.2. Példa.

```
> `&+`:=(x,y)->x+y mod 2;
  0&+0; 0&+1; 1&+0; 1&+1;
```

$$\&+ := (x, y) \rightarrow x+y\ \mathbf{mod}\ 2$$

$$0$$

$$1$$

$$1$$

$$0 \quad (1.1.2.1)$$

1.1.3. Algebrai struktúrák.

1.1.4. Példák.

1.1.5. Mátrixok.

```
> with(linalg);
```

[*BlockDiagonal*, *GramSchmidt*, *JordanBlock*, *LUdecomp*, *QRdecomp*, *Wronskian*, *addcol*, *addrow*, *adj*, *adjoint*, *angle*, *augment*, *backsub*, *band*, *basis*, *bezout*, *blockmatrix*, *charmat*, *charpoly*, *cholesky*, *col*, *coldim*, *colspace*, *colspan*, *companion*, *concat*, *cond*, *copyinto*, *crossprod*, *curl*, *definite*, *delcols*, *delrows*, *det*, *diag*, *diverge*, *dotprod*, *eigenvals*, *eigenvalues*, *eigenvectors*, *eigenvects*, *entermatrix*, *equal*, *exponential*, *extend*, *ffgausselim*, *fibonacci*, *forwardsub*, *frobenius*, *gausselim*, *gaussjord*, *geneqns*, *genmatrix*, *grad*, *hadamard*, *hermite*, *hessian*, *hilbert*, *htranspose*, *ihermite*, *indexfunc*, *innerprod*, *intbasis*, *inverse*, *ismith*, *issimilar*, *iszero*, *jacobian*, *jordan*, *kernel*, *laplacian*, *leastsqrs*, *linsolve*, *matadd*, *matrix*, *minor*, *minpoly*, *mulcol*, *mulrow*, *multiply*, *norm*, *normalize*, *nullspace*, *orthog*, *permanent*, *pivot*, *potential*, *randmatrix*, *randvector*, *rank*, *ratform*, *row*, *rowdim*, *rowspace*, *rowspan*, *rref*, *scalarmul*, *singularvals*, *smith*, *stackmatrix*, *submatrix*, *subvector*, *sumbasis*, *swapcol*, *swaprow*, *sylvester*, *toeplitz*, *trace*, *transpose*, *vandermonde*, *vecpotent*, *vectdim*, *vector*, *wronskian*] (1.1.5.1)

```
> a:=matrix(2,3,[x,cc,Alma,3,5]);
```

$$a := \begin{bmatrix} x & cc & Alma \\ 3 & 5 & a_{2,\,3} \end{bmatrix} \tag{1.1.5.2}$$

```
> a[2,3]:=8;print(a);
```

$$a_{2,\,3} := 8$$

$$\begin{bmatrix} x & cc & Alma \\ 3 & 5 & 8 \end{bmatrix} \tag{1.1.5.3}$$

```
> b:=matrix([[x,y],[9,7]]);
```

$$b := \begin{bmatrix} x & y \\ 9 & 7 \end{bmatrix} \tag{1.1.5.4}$$

```
> c:=matrix(3,2,(i,j)->i+j-1);
```

$$c := \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad (1.1.5.5)$$

```
> d:=extend(c,1,1,0);copyinto(a,d,3,2);
```

$$d := \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & x & cc \\ 0 & 3 & 5 \end{bmatrix} \qquad (1.1.5.6)$$

```
> transpose(c);d:=transpose(%);equal(c,d);
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

$$d := \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$true \qquad (1.1.5.7)$$

```
> a:=matrix(3,2);
```

$$a := array(1..3, 1..2, [\,]) \qquad (1.1.5.8)$$

```
> entermatrix(a);
Warning,  computation interrupted
```

1.1.6. *Műveletek mátrixokkal.*

```
> a:=matrix([[1,0],[0,2]]); b:=matrix([[0,1],[0,0]]);
  c:=matrix([[1,0],[0,1],[0,0]]);
```

$$a := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$b := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$c := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (1.1.6.1)$$

```
> d:=evalm(2*a+3*b); evalm(a&*b); evalm(b&*a); evalm(sin(d));
  evalm(d^2); evalm(c&*a); evalm(a&*c);
```

$$d := \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \sin(2) & \sin(3) \\ 0 & \sin(4) \end{bmatrix}$$

$$\begin{bmatrix} 4 & 18 \\ 0 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

```
Error, (in linalg:-multiply) non matching dimensions for
vector/matrix product
```

▶ *1.1.7. Állítás.*

▼ *1.1.8. Nullmátrix, egységmátrix, inverz mátrix.*

```
> a:=matrix(3,2,0); iszero(a); b:=matrix(2,2,[1,0,0,0]);
  iszero(b);
```

$$a := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

true

$$b := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

false (1.1.8.1)

```
> rowdim(a); coldim(a);
```

$$3$$
$$2 \tag{1.1.8.2}$$

```
> a:=diag(3,4,5);
```

$$a := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \tag{1.1.8.3}$$

```
> krondelta:=(i,j)-> if i=j then 1 else 0 fi;
```

$$krondelta := (i, j) \rightarrow \textbf{if } i = j \textbf{ then } 1 \textbf{ else } 0 \textbf{ end if} \tag{1.1.8.4}$$

```
> a:=matrix(3,3,krondelta);
```

$$a := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1.8.5}$$

```
> b:=matrix(3,2,krondelta);
```

$$b := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{1.1.8.6}$$

```
> c:=hilbert(3); d:=inverse(c); evalm(c&*d); evalm(d&*c);
```

$$c := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$d := \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1.1.8.7)

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad (1.1.8.7)$$

```
> inverse(b);
Error, (in linalg:-inverse) expecting a square matrix
> a:=matrix([[2,4],[1,2]]); inverse(a);
```

$$a := \begin{bmatrix}2 & 4\\1 & 2\end{bmatrix}$$

```
Error, (in linalg:-inverse) singular matrix
```

1.1.9. Elemi sor- és oszlopműveletek.

```
> a:=matrix([[1,2,3],[4,5,6]]);
```

$$a := \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} \quad (1.1.9.1)$$

```
> swaprow(a,1,2);
```

$$\begin{bmatrix}4 & 5 & 6\\1 & 2 & 3\end{bmatrix} \quad (1.1.9.2)$$

```
> swapcol(a,2,3);
```

$$\begin{bmatrix}1 & 3 & 2\\4 & 6 & 5\end{bmatrix} \quad (1.1.9.3)$$

```
> addrow(a,1,2,-4);
```

$$\begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\end{bmatrix} \quad (1.1.9.4)$$

```
> addcol(a,2,3,-3/2);
```

$$\begin{bmatrix}1 & 2 & 0\\4 & 5 & -\frac{3}{2}\end{bmatrix} \quad (1.1.9.5)$$

1.1.10. Gauss-féle kiküszöbölés.

```
> x:='x'; y:='y'; z:='z';
  eqns:={x+y/2+z/3=1,x/2+y/3+z/4=7/12,x/3+y/4+z/5=13/30};
```

$$x := x$$

$$y := y$$

$$z := z$$

$$eqns := \left\{x + \frac{1}{2}\, y + \frac{1}{3}\, z = 1,\ \frac{1}{2}\, x + \frac{1}{3}\, y + \frac{1}{4}\, z = \frac{7}{12},\ \frac{1}{3}\, x + \frac{1}{4}\, y + \frac{1}{5}\, z = \frac{13}{30}\right\} \quad (1.1.10.1)$$

```
> solve(eqns);
```

$$\{z = 3,\ x = 1,\ y = -2\} \quad (1.1.10.2)$$

```
> a:=genmatrix(eqns,[x,y,z]);
```

$$a := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \quad (1.1.10.3)$$

```
> a:=genmatrix(eqns,[x,y,z],'b'); print(b);
```

$$a := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{12} & \frac{13}{30} \end{bmatrix} \quad (1.1.10.4)$$

```
> geneqns(a,[x,y,z],b);
```

$$\left\{x + \frac{1}{2}\, y + \frac{1}{3}\, z = 1,\ \frac{1}{2}\, x + \frac{1}{3}\, y + \frac{1}{4}\, z = \frac{7}{12},\ \frac{1}{3}\, x + \frac{1}{4}\, y + \frac{1}{5}\, z = \frac{13}{30}\right\} \quad (1.1.10.5)$$

```
> c:=augment(a,b);
```

$$c := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{13}{30} \end{bmatrix} \quad (1.1.10.6)$$

```
> genmatrix(eqns,[x,y,z],flag);
```

(1.1.10.7)

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{13}{30} \end{bmatrix} \tag{1.1.10.7}$$

1.1.11. Példák.

```
> c1:=pivot(c,1,1,2..3);
```

$$c1 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{4}{45} & \frac{1}{10} \end{bmatrix} \tag{1.1.11.1}$$

```
> c2:=pivot(c1,2,2,3..3);
```

$$c2 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} & \frac{1}{60} \end{bmatrix} \tag{1.1.11.2}$$

```
> gausselim(c,3);
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} & \frac{1}{60} \end{bmatrix} \tag{1.1.11.3}$$

```
> backsub(c2);
```

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \tag{1.1.11.4}$$

```
> d:=copy(c); d[3,3]:=7/36; print(d);
```

$$d := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{13}{30} \end{bmatrix}$$

$$d_{3,\,3} := \frac{7}{36}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{7}{36} & \frac{13}{30} \end{bmatrix} \qquad (1.1.11.5)$$

```
> d1:=pivot(d,1,1,2..3);
```

$$d1 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{10} \end{bmatrix} \qquad (1.1.11.6)$$

```
> d2:=pivot(d1,2,2,3..3);
```

$$d2 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 0 & \frac{1}{60} \end{bmatrix} \qquad (1.1.11.7)$$

```
> backsub(d2);
Error, (in linalg:-backsub) inconsistent system
> e:=copy(d); e[3,4]:=5/12; print(e);
```

$$e := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{7}{36} & \frac{13}{30} \end{bmatrix}$$

$$e_{3,\,4} := \frac{5}{12}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{7}{36} & \frac{5}{12} \end{bmatrix} \quad (1.1.11.8)$$

```
> e1:=pivot(e,1,1,2..3);
```

$$e1 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \end{bmatrix} \quad (1.1.11.9)$$

```
> e2:=pivot(e1,2,2,3..3);
```

$$e2 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.1.11.10)$$

```
> backsub(e2);
```

$$\begin{bmatrix} \frac{1}{2} + \frac{1}{6}\_t_1 & 1 - \_t_1 & \_t_1 \end{bmatrix} \quad (1.1.11.11)$$

```
> evalm(randmatrix(3,4)+I*randmatrix(3,4)); gausselim(%);
  backsub(%);
```

$$\begin{bmatrix} 22+62\,\mathrm{I} & -55-82\,\mathrm{I} & -94+80\,\mathrm{I} & 87-44\,\mathrm{I} \\ -56+71\,\mathrm{I} & -17\,\mathrm{I} & -62-75\,\mathrm{I} & 97-10\,\mathrm{I} \\ -73-7\,\mathrm{I} & -4-40\,\mathrm{I} & -83+42\,\mathrm{I} & -10-50\,\mathrm{I} \end{bmatrix}$$

$$\left[[-73-7\,I, -4-40\,I, -83+42\,I, -10-50\,I], \left[0, \frac{118682}{2689}+\frac{14957}{2689}\,I, -\frac{269533}{5378}-\frac{1016897}{5378}\,I, \frac{418163}{2689}+\frac{35010}{2689}\,I\right], \left[0, 0, \frac{7738997}{36826}-\frac{8653977}{36826}\,I, \frac{1185088861}{5321357}+\frac{1475940899}{5321357}\,I\right] \right]$$

$$\left[\frac{306082322429}{1057741824457}-\frac{35197922369}{1057741824457}\,I, -\frac{1414384720421}{1057741824457}+\frac{989229441330}{1057741824457}\,I, -\frac{195587870162}{1057741824457}+\frac{1177322214500}{1057741824457}\,I\right] \tag{1.1.11.12}$$

1.1.12. Módosított Gauss-féle kiküszöbölés.

```
> print(c);
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{13}{30} \end{bmatrix} \tag{1.1.12.1}$$

```
> c1:=pivot(c,1,1,2..3);
```

$$c1 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{4}{45} & \frac{1}{10} \end{bmatrix} \tag{1.1.12.2}$$

```
> mulrow(c1,2,12); c2:=pivot(%,2,2,3..3);
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & \frac{1}{12} & \frac{4}{45} & \frac{1}{10} \end{bmatrix}$$

$$c2 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{180} & \frac{1}{60} \end{bmatrix} \qquad (1.1.12.3)$$

```
> c3:=mulrow(c2,3,180);
```

$$c3 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad (1.1.12.4)$$

```
> backsub(c3);
```

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \qquad (1.1.12.5)$$

1.1.13. Mátrixinverzió Gauss-féle kiküszöböléssel.

```
> a:=hilbert(3); b:=diag(1,1,1); c:=augment(a,b);
```

$$a := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$b := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1.1.13.1)

$$c := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix} \quad (1.1.13.1)$$

```
> cc:=gausselim(c,3);
```

$$cc := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix} \quad (1.1.13.2)$$

```
> aa:=submatrix(cc,1..3,1..3); bb:=submatrix(cc,1..3,4..6);
```

$$aa := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{bmatrix}$$

$$bb := \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{6} & -1 & 1 \end{bmatrix} \quad (1.1.13.3)$$

```
> backsub(aa,bb); inverse(a);
```

$$\begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \quad (1.1.13.4)$$

```
> linsolve(a,b);
```

$$\begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \quad (1.1.13.5)$$

1.1.14. Gauss–Jordan kiküszöbölés.

```
> print(c);
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix} \quad (1.1.14.1)$$

```
> c1:=pivot(c,1,1);
```

$$c1 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix} \quad (1.1.14.2)$$

```
> mulrow(c1,2,12); c2:=pivot(%,2,2);
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$c2 := \begin{bmatrix} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix} \quad (1.1.14.3)$$

```
> mulrow(c2,3,180); c3:=pivot(%,3,3);
```

$$\begin{bmatrix} 1 & 0 & -\frac{1}{6} & 4 & -6 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

$$c3 := \begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix} \quad (1.1.14.4)$$

```
> gaussjord(c,3);
```

$$\begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix} \quad (1.1.14.5)$$

*1.1.15. A műveletigények összehasonlítása.

*1.1.16. A háromszögmátrixok módszere.

```
> u:=LUdecomp(a,P='p',L='l');
```

$$u := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{bmatrix} \quad (1.1.16.1)$$

```
> print(p); print(l); evalm(p&*l&*u);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix}$$

(1.1.16.2)

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \quad (1.1.16.2)$$

```
> forwardsub(l,b); backsub(u,%);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{6} & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \quad (1.1.16.3)$$

*1.1.17. Numerikus megjegyzések.

1.1.18. Vektortér.

1.1.19. Példák.

```
> v:=vector(3,[5,6]);
```

$$v := \begin{bmatrix} 5 & 6 & v_3 \end{bmatrix} \quad (1.1.19.1)$$

```
> v[3]:=7; print(v);
```

$$v_3 := 7$$

$$\begin{bmatrix} 5 & 6 & 7 \end{bmatrix} \quad (1.1.19.2)$$

```
> u:=vector(3,i->i+1);
```

$$u := \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \quad (1.1.19.3)$$

```
> evalm(2*u+v);
```

$$\begin{bmatrix} 9 & 12 & 15 \end{bmatrix} \quad (1.1.19.4)$$

```
> w:=vector(4,0); vectdim(w);
```

$$w := \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$4 \quad (1.1.19.5)$$

```
> w:=randvector(3);
```

$$w := \begin{bmatrix} 23 & 75 & -92 \end{bmatrix} \quad (1.1.19.6)$$

```
> type(w,'vector'); type(w,'matrix');
```

true

$$false \qquad (1.1.19.7)$$

1.1.20 Altér.

1.1.21. Példák.

1.1.22. Lineáris kombináció.

```
> evalm(-87*z-7/3*u+16/3*v);
```

$$\begin{bmatrix} 22-87\,z & 25-87\,z & 28-87\,z \end{bmatrix} \qquad (1.1.22.1)$$

1.1.23. Megjegyzés.

```
> z:=vector([1,0,0]); a:=augment(z,u,v,w);
```

$$z := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$a := \begin{bmatrix} 1 & 2 & 5 & 23 \\ 0 & 3 & 6 & 75 \\ 0 & 4 & 7 & -92 \end{bmatrix} \qquad (1.1.23.1)$$

```
> b:=stackmatrix(u,v,w,z);
```

$$b := \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 23 & 75 & -92 \\ 1 & 0 & 0 \end{bmatrix} \qquad (1.1.23.2)$$

```
> col(a,2);
```

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \qquad (1.1.23.3)$$

```
> col(a,2..3);
```

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} \qquad (1.1.23.4)$$

```
> row(a,2);
```

$$\begin{bmatrix} 0 & 3 & 6 & 75 \end{bmatrix} \qquad (1.1.23.5)$$

```
> row(a,2..3);
```

$$\begin{bmatrix} 0 & 3 & 6 & 75 \end{bmatrix}, \begin{bmatrix} 0 & 4 & 7 & -92 \end{bmatrix} \qquad (1.1.23.6)$$

```
> subvector(b,3,2..3); subvector(b,1..3,2);
```

$$\begin{bmatrix} 75 & -92 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 75 \end{bmatrix} \qquad (1.1.23.7)$$

1.1.24. Tétel.

*1.1.25. Tétel.

1.1.26. Generátorrendszer, dimenzió.

1.1.27. Lineáris függetlenség és függőség.

1.1.28. Tétel.

1.1.29. Bázis.

```
> basis([z,u,v,w]); basis([u,v,w,z]);
```

$$[z, u, v]$$

$$[u, v, w] \quad (1.1.29.1)$$

```
> intbasis([z,u],[u,v,w]); intbasis([z,u],[v,w]); intbasis(
  [z,u],[v]);
```

$$\{[-937 \quad -1077 \quad -1436], [2 \quad 3 \quad 4]\}$$

$$\{[-937 \quad -1077 \quad -1436]\}$$

$$\{ \} \quad (1.1.29.2)$$

```
> sumbasis([z,u],[v,w]);
```

$$[z, u, v] \quad (1.1.29.3)$$

1.1.30. Tétel.

1.1.31. Következmény.

1.1.32. Következmény.

1.1.33. Következmény.

1.1.34. Következmény.

1.1.35. Következmény.

1.1.36. Következmény.

1.1.37. Következmény.

1.1.38. Rang.

```
> basis(a,colspace); basis(a,rowspace);
```

$$[[1 \quad 0 \quad 0], [2 \quad 3 \quad 4], [5 \quad 6 \quad 7]]$$

$$[[1 \quad 2 \quad 5 \quad 23], [0 \quad 3 \quad 6 \quad 75], [0 \quad 4 \quad 7 \quad -92]] \quad (1.1.38.1)$$

```
> print(a); backsub(gausselim(a));
```

$$\begin{bmatrix} 1 & 2 & 5 & 23 \\ 0 & 3 & 6 & 75 \\ 0 & 4 & 7 & -92 \end{bmatrix}$$

$$[-219 \quad -359 \quad 192] \quad (1.1.38.2)$$

1.1.39. Megjegyzés.

```
> rank(a); gausselim(a,'ra');ra;
```

$$3$$

$$\begin{bmatrix} 1 & 2 & 5 & 23 \\ 0 & 3 & 6 & 75 \\ 0 & 0 & -1 & -192 \end{bmatrix}$$

$$3 \qquad (1.1.39.1)$$

```
> rank(b); gausselim(b,'rb');rb;
```

$$3$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -\frac{3}{2} & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$3 \qquad (1.1.39.2)$$

▶ *1.1.40. Tétel.*

▶ *1.1.41. Következmény.*

▶ *1.1.42. Tétel.*

▶ *1.1.43. Koordináták.*

▶ *1.1.44. Izomorfia.*

▶ *1.1.45. Tétel.*

▶ **1.1.46. Vektorterek direkt összege.*

▶ **1.1.48. Következmény.*

▶ **1.1.49. Példa.*

▶ *1.1.50. Affin sokaságok.*

▼ *1.1.51. Tétel.*

```
> x:=vector([1+t,2-t,t-1]); x0:=map2(subs,t=0,x); y:=evalm(x-
x0);
```

$$x := \begin{bmatrix} 1+t & 2-t & t-1 \end{bmatrix}$$

$$x0 := \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

$$y := \begin{bmatrix} t & -t & t \end{bmatrix} \qquad (1.1.50.1)$$

```
> y1:=map2(subs,t=1,y);
```

$$y1 := \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \qquad (1.1.50.2)$$

```
> z:=evalm(x0+t*y1);
```

$$z := \begin{bmatrix} 1+t & 2-t & t-1 \end{bmatrix} \qquad (1.1.50.3)$$

▶ *1.1.52. Következmény.*

1.1.53. Következmény.

```
> x:=vector([1+t1,2-t2,t1+t2-1]); x0=map2(subs,[t1=0,t2=0],x)
;
y:=evalm(x-x0);
```

$$x := \begin{bmatrix} 1+t1 & 2-t2 & t1+t2-1 \end{bmatrix}$$

$$x0 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

$$y := \begin{bmatrix} t1 & -t2 & t1+t2 \end{bmatrix} \quad (1.1.52.1)$$

```
> y1:=map2(subs,[t1=1,t2=0],y); y2:=map2(subs,[t1=0,t2=1],y);
```

$$y1 := \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$y2 := \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \quad (1.1.52.2)$$

```
> z:=evalm(x0+t1*y1+t2*y2);
```

$$z := \begin{bmatrix} 1+t1 & 2-t2 & t1+t2-1 \end{bmatrix} \quad (1.1.52.3)$$

```
> x:=vector([1+t1+t3,2+t2+t3,-1,2]); x0:=map2(subs,[t1=0,t2=
0,t3=0],x);
y:=evalm(x-x0);
```

$$x := \begin{bmatrix} 1+t1+t3 & 2+t2+t3 & -1 & 2 \end{bmatrix}$$

$$x0 := \begin{bmatrix} 1 & 2 & -1 & 2 \end{bmatrix}$$

$$y := \begin{bmatrix} t1+t3 & t2+t3 & 0 & 0 \end{bmatrix} \quad (1.1.52.4)$$

```
> y1:=map2(subs,[t1=1,t2=0,t3=0],y); y2:=map2(subs,[t1=0,t2=
1,t3=0],y);
y3:=map2(subs,[t1=0,t2=0,t3=1],y);
```

$$y1 := \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$y2 := \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$y3 := \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \quad (1.1.52.5)$$

```
> basis([y1,y2,y3]);
```

$$[y1, y2] \quad (1.1.52.6)$$

```
> z:=evalm(x0+u1*y1+u2*y2);
```

$$z := \begin{bmatrix} 1+u1 & 2+u2 & -1 & 2 \end{bmatrix} \quad (1.1.52.7)$$

1.1.54. Affin sokaság dimenziója.

1.2. Lineáris leképezések

```
> restart;with(linalg);
```

[*BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col,* (1.2.1) *coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian*]

▶ ***1.2.1. Lineáris leképezések.***

▶ ***1.2.2. Példák lineáris leképezésekre.***

▶ ***1.2.3. Tétel.***

▶ ***1.2.4. Lineáris leképezések összege és skalárszorosa.***

▶ ***1.2.5. Tétel.***

▶ ***1.2.6. Lineáris leképezések szorzata.***

▶ ***1.2.7. Tétel.***

▶ ***1.2.8. Következmény.***

▶ ***1.2.9. Tétel.***

▶ ***1.2.10. Következmény.***

▶ ***1.2.11. Következmény.***

▼ ***1.2.12. Vektorrendszer és leképezés mátrixa.***

```
> a:=matrix([[1,1,2,5,0],[0,1,1,3,-1],[1,0,1,2,1]]);
```

$$a := \begin{bmatrix} 1 & 1 & 2 & 5 & 0 \\ 0 & 1 & 1 & 3 & -1 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix} \tag{1.2.12.1}$$

```
> colspace(a); rank(a); kernel(a); coldim(a);
```

$$\{[1 \quad 0 \quad 1], [0 \quad 1 \quad -1]\}$$

$$2$$

$$\{[-1 \;\; -1 \;\; 1 \;\; 0 \;\; 0], [-2 \;\; -3 \;\; 0 \;\; 1 \;\; 0], [-1 \;\; 1 \;\; 0 \;\; 0 \;\; 1]\}$$

$$5 \qquad (1.2.12.2)$$

1.2.13. Tétel.

```
> a:=matrix([[1,2,3],[2,3,4],[3,4,5]]); x:=vector([1,0,1]);
  multiply(a,x);
```

$$a := \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$x := [1 \;\; 0 \;\; 1]$$

$$[4 \;\; 6 \;\; 8] \qquad (1.2.13.1)$$

1.2.14. Következmény.

1.2.15. Tétel.

1.2.16. Áttérésmátrix.

1.2.17. Példák: egyszerű bázistranszformációk.

1.2.18. Tétel.

1.2.19. Következmény.

1.2.20. Következmény.

1.2.21. Ekvivalens transzformációk, ekvivalens mátrixok.

```
> t:=matrix([[1,1,0],[0,1,1],[1,0,1]]); rank(t);
```

$$t := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$3 \qquad (1.2.21.1)$$

```
> b:=evalm(inverse(t)&*a&*t);
```

$$b := \begin{bmatrix} 3 & \frac{5}{2} & \frac{7}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \\ 5 & \frac{9}{2} & \frac{11}{2} \end{bmatrix} \qquad (1.2.21.2)$$

```
> s:=matrix([[1,-2,0],[0,1,-2],[-2,0,1]]); rank(s);
```

$$s := \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$3 \quad (1.2.21.3)$$

```
> evalm(inverse(s)&*evalm(a&*t));
```

$$\begin{bmatrix} -\frac{48}{7} & -\frac{41}{7} & -\frac{55}{7} \\ -\frac{38}{7} & -\frac{31}{7} & -\frac{45}{7} \\ -\frac{40}{7} & -\frac{33}{7} & -\frac{47}{7} \end{bmatrix} \quad (1.2.21.4)$$

```
> augment(s,evalm(a&*t)); gaussjord(%,3);
```

$$\begin{bmatrix} 1 & -2 & 0 & 4 & 3 & 5 \\ 0 & 1 & -2 & 6 & 5 & 7 \\ -2 & 0 & 1 & 8 & 7 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{48}{7} & -\frac{41}{7} & -\frac{55}{7} \\ 0 & 1 & 0 & -\frac{38}{7} & -\frac{31}{7} & -\frac{45}{7} \\ 0 & 0 & 1 & -\frac{40}{7} & -\frac{33}{7} & -\frac{47}{7} \end{bmatrix} \quad (1.2.21.5)$$

1.2.22. Lineáris formák.

```
> f:=matrix(1,4); x:=vector(4); multiply(f,x);
```

$$f := array(1..1, 1..4, [\,])$$

$$x := array(1..4, [\,])$$

$$\left[f_{1,1}\, x_1 + f_{1,2}\, x_2 + f_{1,3}\, x_3 + f_{1,4}\, x_4\right] \quad (1.2.22.1)$$

**1.2.23. Tétel.*

1.2.24. Bilineáris leképezések.

```
> a:=matrix(2,3); x:=vector(2); y:=vector(3); innerprod(x,a,
  y);
  evalm(x&*a&*y);
```

$$a := array(1..2, 1..3, [\,])$$

$$x := array(1..2, [\,])$$

$$y := array(1..3, [\,])$$

$$y_1\, x_1\, a_{1,\,1} + y_1\, x_2\, a_{2,\,1} + y_2\, x_1\, a_{1,\,2} + y_2\, x_2\, a_{2,\,2} + y_3\, x_1\, a_{1,\,3} + y_3\, x_2\, a_{2,\,3}$$

$$\left(x_1\, a_{1,\,1} + x_2\, a_{2,\,1}\right) y_1 + \left(x_1\, a_{1,\,2} + x_2\, a_{2,\,2}\right) y_2 + \left(x_1\, a_{1,\,3} + x_2\, a_{2,\,3}\right) y_3 \quad (1.2.24.1)$$

```
> x:=vector(3); innerprod(x,y);
```

$$x := array(1..3, [\,])$$

$$x_1\, y_1 + x_2\, y_2 + x_3\, y_3 \quad (1.2.24.2)$$

▶ *1.2.25. Tétel.*

▶ *1.2.26. Következmény.*

▼ *1.2.27. Kvadratikus leképezések.*

```
> b:=matrix([[1,2,3],[2,3,4],[5,6,7]]); q:=innerprod(x,b,x);
```

$$b := \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

$$q := x_1^2 + 4\, x_1\, x_2 + 8\, x_1\, x_3 + 3\, x_2^2 + 10\, x_2\, x_3 + 7\, x_3^2 \quad (1.2.27.1)$$

▼ *1.2.28. Tétel.*

```
> s:=evalm(1/2*b+1/2*transpose(b)); a:=evalm(1/2*b-1/2*
  transpose(b));
```

$$s := \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

$$a := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad (1.2.28.1)$$

```
> innerprod(x,s,x); innerprod(x,a,x);
```

$$x_1^2 + 4\, x_1\, x_2 + 8\, x_1\, x_3 + 3\, x_2^2 + 10\, x_2\, x_3 + 7\, x_3^2$$

$$0 \quad (1.2.28.2)$$

▼ *1.2.29. Példa: vektori szorzat.*

```
> a:=matrix([[0,k,-j],[-k,0,i],[j,-i,0]]); b:=innerprod(x,a,
  y);
```

$$a := \begin{bmatrix} 0 & k & -j \\ -k & 0 & i \\ j & -i & 0 \end{bmatrix}$$

$$b := -y_1 x_2 k + y_1 x_3 j + y_2 x_1 k - y_2 x_3 i - y_3 x_1 j + y_3 x_2 i \quad (1.2.29.1)$$

```
> i:=vector([1,0,0]); j:=vector([0,1,0]); k:=vector([0,0,1]);
  evalm(b);
```

$$i := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$
$$j := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$
$$k := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -y_2 x_3 + y_3 x_2 & y_1 x_3 - y_3 x_1 & -y_1 x_2 + y_2 x_1 \end{bmatrix} \quad (1.2.29.2)$$

```
> crossprod(x,y);
```

$$\begin{bmatrix} -y_2 x_3 + y_3 x_2 & y_1 x_3 - y_3 x_1 & -y_1 x_2 + y_2 x_1 \end{bmatrix} \quad (1.2.29.3)$$

1.2.30. Tétel: szimmetrikus és alternáló bilineáris leképezések mátrixa.

1.2.31. Szimmetrikus bilineáris formák átlós alakra hozása.

```
> print(s); addcol(s,1,2,-2); addrow(%,1,2,-2); addcol(%,1,3,
  -4);
  s1:=addrow(%,1,3,-4);
```

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & -1 & 5 \\ 4 & -3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & -1 & -3 \\ 4 & -3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 4 & -3 & -9 \end{bmatrix}$$

(1.2.31.1)

$$s1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & -3 & -9 \end{bmatrix} \quad (1.2.31.1)$$

```
> addcol(s1,2,3,-3); s2:=addrow(%,2,3,-3);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \end{bmatrix}$$

$$s2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.2.31.2)$$

1.2.32. Megjegyzés.

**1.2.33. Antiszimmetrikus bilineáris formák átlós alakra hozása.*

1.2.34. Definit és indefinit kvadratikus formák.

```
> definite(s,'positive_def'); definite(s,'positive_semidef');
  definite(s,'negative_semidef');
```

false

false

false (1.2.34.1)

```
> princaxis:=proc(b) local a,i,j,k,n,x;
  if not(type(b,'matrix'(rational,square))
    or type(b,'matrix'(float,square))) then ERROR(`Invalid
  argument`) fi;
  n:=coldim(b);
  for i to n do for j from i+1 to n do
    if b[i,j]<>b[j,i] then ERROR(`Invalid argument`) fi;
  od; od;
  a:=copy(b);
  for k to n do
    if a[k,k]=0 then
      for j from k+1 to n do if a[j,j]<>0 then break fi; od;
      if j<=n then a:=swapcol(a,k,j); a:=swaprow(a,k,j); fi;
    fi;
    if a[k,k]=0 then
      for j from k+1 to n do if a[k,j]<>0 then break fi; od;
      if j<=n then a:=addcol(a,j,k); a:=addrow(a,j,k); fi;
    fi;
    if a[k,k]<>0 then
```

```
      for j from k+1 to n do
        x:=-a[k,j]/a[k,k]; a:=addcol(a,k,j,x);a:=addrow(a,k,
  j,x);
      od;
    fi;
  od; evalm(a); end;
```

$princaxis := \mathbf{proc}(b)$ (1.2.34.2)

```
    local a, i, j, k, n, x;
    if not (type(b, 'linalg:-matrix'(rational,
    square)) or type(b, 'linalg:-matrix'(float, square))) then
        ERROR(Invalid argument)
    end if;
    n := linalg:-coldim(b);
    for i to n do
        for j from i + 1 to n do
            if b[i, j] <> b[j, i] then
                ERROR(Invalid argument)
            end if
        end do
    end do;
    a := copy(b);
    for k to n do
        if a[k, k] = 0 then
            for j from k + 1 to n do
                if a[j, j] <> 0 then
                    break
                end if
            end do;
            if j <= n then
                a := linalg:-swapcol(a, k, j);
                a := linalg:-swaprow(a, k, j)
            end if
        end if;
        if a[k, k] = 0 then
```

```
            for j from k + 1 to n do
                if a[k, j] <> 0 then
                    break
                end if
            end do;
            if j <= n then
                a := linalg:-addcol(a, j, k);
                a := linalg:-addrow(a, j, k)
            end if
        end if;
        if a[k, k] <> 0 then
            for j from k + 1 to n do
                x := -a[k, j] / a[k, k];
                a := linalg:-addcol(a, k, j, x);
                a := linalg:-addrow(a, k, j, x)
            end do
        end if
    end do;
    evalm(a)
end proc
> princaxis(s);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (1.2.34.3)$$

▶ ***1.2.35. Sylvester-féle tehetetlenségi tétel.***

▶ ****1.2.36. Konjugált bilineáris leképezések.***

▶ ****1.2.37. Tétel.***

▶ ****1.2.38. Következmény.***

▼ ****1.2.39. Hermite-formák.***

▶ ****1.2.40. Multilineáris leképezések.***

▶ ****1.2.41. Tétel.***

▶ ****1.2.42. Példa.***

*1.2.43. Tenzorok, tenzorszorzat.
*1.2.44. Segédtétel.
*1.2.45. Tenzorok koordinátáinak transzformációja.
*1.2.46. Einstein-konvenció.
1.2.47. Permutációk.
1.2.48. Tétel.
1.2.49. Következmény.
1.2.50. Terület, térfogat és determináns.
1.2.51. Tétel.
1.2.52. Következmény.
*1.2.53. Következmény.
1.2.54. Következmény.
1.2.55. Mátrix determinánsa.

```
> a:=matrix(4,4); det(a);
```

$$a := array(1..4, 1..4, [\,])$$

$$a_{1,1}a_{2,2}a_{3,3}a_{4,4} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,1}a_{3,2}a_{2,3}a_{4,4} + a_{1,1}a_{3,2}a_{2,4}a_{4,3} + a_{1,1}a_{4,2}a_{2,3}a_{3,4} - a_{1,1}a_{4,2}a_{2,4}a_{3,3} - a_{2,1}a_{1,2}a_{3,3}a_{4,4} + a_{2,1}a_{1,2}a_{3,4}a_{4,3} + a_{2,1}a_{3,2}a_{1,3}a_{4,4} - a_{2,1}a_{3,2}a_{1,4}a_{4,3} - a_{2,1}a_{4,2}a_{1,3}a_{3,4} + a_{2,1}a_{4,2}a_{1,4}a_{3,3} + a_{3,1}a_{1,2}a_{2,3}a_{4,4} - a_{3,1}a_{1,2}a_{2,4}a_{4,3} - a_{3,1}a_{2,2}a_{1,3}a_{4,4} + a_{3,1}a_{2,2}a_{1,4}a_{4,3} + a_{3,1}a_{4,2}a_{1,3}a_{2,4} - a_{3,1}a_{4,2}a_{1,4}a_{2,3} - a_{4,1}a_{1,2}a_{2,3}a_{3,4} + a_{4,1}a_{1,2}a_{2,4}a_{3,3} + a_{4,1}a_{2,2}a_{1,3}a_{3,4} - a_{4,1}a_{2,2}a_{1,4}a_{3,3} - a_{4,1}a_{3,2}a_{1,3}a_{2,4} + a_{4,1}a_{3,2}a_{1,4}a_{2,3} \tag{1.2.55.1}$$

1.2.56. Tétel.

```
> a:=matrix(1,1); det(a);
```

$$a := array(1..1, 1..1, [\,])$$

$$a_{1,1} \tag{1.2.56.1}$$

```
> a:=matrix(2,2); det(a);
```

$$a := array(1..2, 1..2, [\,])$$

$$a_{1,\,1}\,a_{2,\,2} - a_{1,\,2}\,a_{2,\,1} \tag{1.2.56.2}$$

```
> a:=matrix(3,3); det(a);
```

$$a := array(1..3, 1..3, [\,])$$

$$a_{1,\,1}\,a_{2,\,2}\,a_{3,\,3} - a_{1,\,1}\,a_{2,\,3}\,a_{3,\,2} - a_{2,\,1}\,a_{1,\,2}\,a_{3,\,3} + a_{2,\,1}\,a_{1,\,3}\,a_{3,\,2} + a_{3,\,1}\,a_{1,\,2}\,a_{2,\,3} - a_{3,\,1}\,a_{1,\,3}\,a_{2,\,2} \tag{1.2.56.3}$$

1.2.57. Tétel.

1.2.58. Következmény.

1.2.59. Részmátrix, aldetermináns.

1.2.60. Tétel.

1.2.61. Tétel.

1.2.62. Tétel.

```
> a:=hilbert(3); det(a); gausselim(a,'ra','da'); da;
```

$$a := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$\frac{1}{2160}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & \frac{1}{180} \end{bmatrix}$$

$$\frac{1}{2160} \tag{1.2.62.1}$$

1.2.63. Kifejtési tétel.

```
> delcols(a,2..3);delrows(a,1..1);
```

$$\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \qquad (1.2.63.1)$$

```
> delcols(delrows(a,1..1),1..1);A11:=minor(a,1,1);A12:=minor
  (a,1,2);A13:=minor(a,1,3);
```

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$A11 := \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$A12 := \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{5} \end{bmatrix}$$

$$A13 := \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \qquad (1.2.63.2)$$

```
> a[1,1]*det(A11)-a[1,2]*det(A12)+a[1,3]*det(A13);
```

$$\frac{1}{2160} \qquad (1.2.63.3)$$

▶ ***1.2.64. Lineáris transzformáció determinánsa.***

▶ ***1.2.65. Determinánsok szorzástétele.***

▶ ***1.2.66. Determinánsok szorzástétele.***

▶ ***1.2.67. Megjegyzés: lineáris transzformáció determinánsa.***

▶ ***1.2.68. Tétel.***

1.2.69. Tétel.

```
> b:=adjoint(a); evalm(a&*b);
```

$$b := \begin{bmatrix} \frac{1}{240} & -\frac{1}{60} & \frac{1}{72} \\ -\frac{1}{60} & \frac{4}{45} & -\frac{1}{12} \\ \frac{1}{72} & -\frac{1}{12} & \frac{1}{12} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2160} & 0 & 0 \\ 0 & \frac{1}{2160} & 0 \\ 0 & 0 & \frac{1}{2160} \end{bmatrix} \quad (1.2.69.1)$$

1.2.70. Lineáris egyenletek.

```
> a:=matrix([[1,1,2,5,0],[0,1,1,3,-1],[1,0,1,2,1]]);
```

$$a := \begin{bmatrix} 1 & 1 & 2 & 5 & 0 \\ 0 & 1 & 1 & 3 & -1 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix} \quad (1.2.70.1)$$

```
> colspace(a); rank(a); kernel(a); coldim(a);
```

$$\{[1 \quad 0 \quad 1], [0 \quad 1 \quad -1]\}$$

$$2$$

$$\{[-1 \quad -1 \quad 1 \quad 0 \quad 0], [-2 \quad -3 \quad 0 \quad 1 \quad 0], [-1 \quad 1 \quad 0 \quad 0 \quad 1]\}$$

$$5 \quad (1.2.70.2)$$

```
> b:=vector([0,0,0]); linsolve(a,b);
```

$$b := [0 \quad 0 \quad 0]$$

$$[-\_t_1 - 2\,\_t_2 - \_t_3 \quad -\_t_1 - 3\,\_t_2 + \_t_3 \quad \_t_1 \quad \_t_2 \quad \_t_3] \quad (1.2.70.3)$$

```
> b:=vector([1,1,1]); linsolve(a,b);
```

$$b := [1 \quad 1 \quad 1] \quad (1.2.70.4)$$

1.2.71. Állítás.

1.2.72. A szuperpozíció elve.

1.2.73. Következmény.

1.2.74. Következmény.

1.2.75. Következmény.

1.2.76. Következmény.

1.2.77. Következmény.

```
> b:=vector([0,0,0]); linsolve(a,b);
```

$$b := \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\_t_1 - 2\,\_t_2 - \_t_3 & -\_t_1 - 3\,\_t_2 + \_t_3 & \_t_1 & \_t_2 & \_t_3 \end{bmatrix} \quad (1.2.77.1)$$

```
> b:=vector([1,1,0]); linsolve(a,b);
```

$$b := \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\_t_2 - 2\,\_t_1 - \_t_3 & 1 - \_t_2 - 3\,\_t_1 + \_t_3 & \_t_2 & \_t_1 & \_t_3 \end{bmatrix} \quad (1.2.77.2)$$

1.2.78. Invariáns alterek.

1.2.79. Sajátérték, sajátvektor, sajátaltér.

1.2.80. Karakterisztikus polinom.

```
> a:=matrix([[1,2,I],[2,-I,3],[-1,2*I,4]]);
  evalm(-charmat(a,lambda)); det(%); (-1)^3*charpoly(a,
  lambda);
```

$$a := \begin{bmatrix} 1 & 2 & I \\ 2 & -I & 3 \\ -1 & 2\,I & 4 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda + 1 & 2 & I \\ 2 & -\lambda - I & 3 \\ -1 & 2\,I & -\lambda + 4 \end{bmatrix}$$

$$-\lambda^3 + 5\,\lambda^2 - I\,\lambda^2 + 10\,I\,\lambda - 25 - 10\,I$$

$$-\lambda^3 + 5\,\lambda^2 - I\,\lambda^2 + 10\,I\,\lambda - 25 - 10\,I \quad (1.2.80.1)$$

1.2.81. Tétel.

```
> evalf(solve(%,lambda)); %[1]+%[2]+%[3]; trace(a);
```

$4.395490370 + 1.351578558\,I,\ 2.434682355 - 1.229735917\,I,$

$-1.830172723 - 1.121842643\,I$

$$5.000000002 - 1.000000002\,I$$

$$5 - I \quad (1.2.81.1)$$

```
> evalf(eigenvals(a));
```

$4.395490370 + 1.351578558\,I,\ 2.434682355 - 1.229735917\,I,$ (1.2.81.2)

$-1.830172723 - 1.121842643\,I$

```
> evalf(eigenvects(a));
```

$$[4.395490370 + 1.351578558\,I, 1., \{[[1., 1.954505136 - 0.6951699149\,I, 2.741918387 + 0.513519901\,I]]\}], [2.434682355 - 1.229735917\,I, 1., \{[[1., 0.5074154780 - 0.4694586252\,I, -0.2908186661 - 0.4198513982\,I]]\}], [-1.830172723 - 1.121842643\,I, 1., \{[[1., -1.211920613 - 0.5853714612\,I, 0.0489002809 + 0.4063314964\,I]]\}] \tag{1.2.81.3}$$

1.2.82. *Algebrai multiplicitás.*

```
> matrix([[1,1,0],[0,1,0],[0,0,5]]); eigenvects(%);
```

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$[5, 1, \{[0 \quad 0 \quad 1]\}], [1, 2, \{[1 \quad 0 \quad 0]\}] \tag{1.2.82.1}$$

```
> a:=matrix([[1,1,0],[0,1,0],[0,0,1]]); eigenvects(%);
```

$$a := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[1, 3, \{[1 \quad 0 \quad 0], [0 \quad 0 \quad 1]\}] \tag{1.2.82.2}$$

```
> matrix([[1,1,0],[0,1,1],[0,0,1]]); eigenvects(%);
```

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[1, 3, \{[1 \quad 0 \quad 0]\}] \tag{1.2.82.3}$$

**1.2.83. Cayley-Hamilton-tétel.*

```
> a:=matrix([[1,0,0],[1,1,0],[0,0,2]]); p:=charpoly(a,lambda)
; expand(p);
L:=PolynomialTools[CoefficientList](p,lambda);
L:=zip((x,y)->x*y,L,[a&^i$i=0..nops(L)-1]);
```

```
p:=convert(L,`+`); evalm(p);
```

$$a := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$p := (\lambda - 1)^2 (\lambda - 2)$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2$$

$$L := [-2, 5, -4, 1]$$

$$L := [-2\, a \mathbin{\&\hat{}}\, 0, 5\, a \mathbin{\&\hat{}}\, 1, -4\, a \mathbin{\&\hat{}}\, 2, a \mathbin{\&\hat{}}\, 3]$$

$$p := -2\, a \mathbin{\&\hat{}}\, 0 + 5\, a \mathbin{\&\hat{}}\, 1 - 4\, a \mathbin{\&\hat{}}\, 2 + a \mathbin{\&\hat{}}\, 3$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (1.2.83.1)$$

1.2.84. Felső háromszög alak.

```
> t:=matrix([[1,0,0],[0,1,2],[0,1,1]]); tm:=inverse(t); b:=
  evalm(tm&*a&*t);
```

$$t := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$tm := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$b := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix} \qquad (1.2.84.1)$$

```
> eigenvects(b);
```

$$[1, 2, \{[0 \quad -1 \quad 1]\}], [2, 1, \{[0 \quad -2 \quad 1]\}] \qquad (1.2.84.2)$$

```
> f11:=vector([0,-2,1]); e11:=vector([1,0,0]); e12:=vector(
  [0,1,0]); e13:=vector([0,0,1]); basis([f11,e11,e12,e13]);
```

$$f11 := [0 \quad -2 \quad 1]$$

$$e11 := [1 \quad 0 \quad 0]$$

$$e12 := [0 \quad 1 \quad 0]$$

$$e13 := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$[f11, e11, e12] \quad (1.2.84.3)$$

```
> t1:=matrix([[0,1,0],[-2,0,1],[1,0,0]]); t1m:=inverse(t1);
bb:=evalm(t1m&*b&*t1);
```

$$t1 := \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$t1m := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$bb := \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (1.2.84.4)$$

```
> b1:=matrix([[1,0],[1,1]]); eigenvects(b1);
```

$$b1 := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\left[1, 2, \left\{\begin{bmatrix} 0 & 1 \end{bmatrix}\right\}\right] \quad (1.2.84.5)$$

```
> f22:=vector([0,1]); e22:=vector([1,0]); e23:=vector([0,1]);
basis([f22,e22,e23]);
```

$$f22 := \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$e22 := \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$e23 := \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$[f22, e22] \quad (1.2.84.6)$$

```
> t2:=matrix([[0,1],[1,0]]); t2m:=inverse(t2); bb1:=evalm
(t2m&*b1&*t2);
```

$$t2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$t2m := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$bb1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (1.2.84.7)$$

```
> b2:=matrix([[1]]); eigenvects(b2);
```

$$b2 := \begin{bmatrix} 1 \end{bmatrix}$$

$$[1, 1, \{[1]\}] \qquad (1.2.84.8)$$

```
> f23:=1*e22; print(f23);
```

$$f23 := e22$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (1.2.84.9)$$

```
> print(f11); f12:=0*e11+1*e12; print(f12); f13:=1*e11+0*e12;
  print(f13);
```

$$\begin{bmatrix} 0 & -2 & 1 \end{bmatrix}$$

$$f12 := e12$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$f13 := e11$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad (1.2.84.10)$$

```
> t:=matrix([[0,0,1],[-2,1,0],[1,0,0]]); tm:=inverse(t);
  print(evalm(tm&*b&*t));
```

$$t := \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$tm := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.2.84.11)$$

▶ ***1.2.85. Következmény.***

▶ ***1.2.86. Segédtétel.***

▼ ***1.2.87. Átlós alak.***

```
> a:=matrix([[2,0,0],[0,1,0],[0,0,3]]);
  t:=matrix([[1,-4,3],[1,-2,2],[3,-3,-2]]); tm:=inverse(t);
  b:=evalm(tm&*a&*t);
```

$$a := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$t := \begin{bmatrix} 1 & -4 & 3 \\ 1 & -2 & 2 \\ 3 & -3 & -2 \end{bmatrix}$$

$$tm := \begin{bmatrix} -\frac{10}{13} & \frac{17}{13} & \frac{2}{13} \\ -\frac{8}{13} & \frac{11}{13} & -\frac{1}{13} \\ -\frac{3}{13} & \frac{9}{13} & -\frac{2}{13} \end{bmatrix}$$

$$b := \begin{bmatrix} \frac{15}{13} & \frac{28}{13} & -\frac{38}{13} \\ -\frac{14}{13} & \frac{51}{13} & -\frac{20}{13} \\ -\frac{15}{13} & \frac{24}{13} & \frac{12}{13} \end{bmatrix} \tag{1.2.87.1}$$

```
> L:=eigenvects(b); b1:=op(1,L[1][3]); b2:=op(1,L[2][3]);
  b3:=op(1,L[3][3]);
  tb:=augment(b1,b2,b3); evalm(inverse(tb)&*b&*tb);
```

$$L := \left[3, 1, \left\{\begin{bmatrix} -2 & 1 & 2 \end{bmatrix}\right\}\right], \left[2, 1, \left\{\begin{bmatrix} \frac{10}{3} & \frac{8}{3} & 1 \end{bmatrix}\right\}\right], \left[1, 1, \left\{\begin{bmatrix} \frac{17}{9} & \frac{11}{9} & 1 \end{bmatrix}\right\}\right]$$

$$b1 := \begin{bmatrix} -2 & 1 & 2 \end{bmatrix}$$

$$b2 := \begin{bmatrix} \frac{10}{3} & \frac{8}{3} & 1 \end{bmatrix}$$

$$b3 := \begin{bmatrix} \frac{17}{9} & \frac{11}{9} & 1 \end{bmatrix}$$

$$tb := \begin{bmatrix} -2 & \frac{10}{3} & \frac{17}{9} \\ 1 & \frac{8}{3} & \frac{11}{9} \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2.87.2}$$

1.3. Belső szorzat

```
> restart;with(linalg);
```

[*BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian*] (1.3.1)

1.3.1. Norma.

1.3.2. Példák.

```
> x:=vector([1,1,1]);
  norm(x,1), norm(x,1.5), norm(x,2), norm(x,3), norm(x,
  infinity), norm(x);
  evalf(%);
```

$$x := [1 \quad 1 \quad 1]$$

$$3,\ 2.080083823,\ \sqrt{3},\ 3^{1/3},\ 1,\ 1$$

$$3.,\ 2.080083823,\ 1.732050808,\ 1.442249570,\ 1.,\ 1. \qquad (1.3.2.1)$$

```
> y:=vector([1,2,3+4*I]);
  norm(y,1), norm(y,1.5), norm(y,2), norm(y,3), norm(y,
  infinity), norm(y);evalf(%);
```

$$y := [1 \quad 2 \quad 3+4\,\mathrm{I}]$$

$$8,\ 6.084571667,\ \sqrt{30},\ 134^{1/3},\ 5,\ 5$$

$$8.,\ 6.084571667,\ 5.477225575,\ 5.117229947,\ 5.,\ 5. \qquad (1.3.2.2)$$

1.3.3. Belső szorzat.

1.3.4. Tétel.

1.3.5. Példa.

```
> dotprod(x,y); dotprod(y,x);
```

$$6-4\,\mathrm{I}$$

$$6+4\,\mathrm{I} \qquad (1.3.5.1)$$

1.3.6. Megjegyzés.

1.3.7. Szögek, euklidészi tér, unitér tér.

```
> y:=vector([1,-1,0]); angle(x,y); angle(y,x);
```

$$y := [1 \quad -1 \quad 0]$$

$$\frac{1}{2}\pi$$

$$\frac{1}{2}\pi \qquad (1.3.7.1)$$

1.3.8. Definíció.

```
> z:=vector([2,3,4]); dotprod(x,y); dotprod(y,z);
```

$$z := [2 \quad 3 \quad 4]$$

$$0$$

$$-1 \qquad (1.3.8.1)$$

1.3.9. Vegyes szorzat.

```
> dotprod(x,crossprod(y,z)); det(augment(x,y,z));
```

$$-3$$

$$-3 \qquad (1.3.9.1)$$

1.3.10. Általánosított Pythagorasz-tétel.

1.3.11. Következmény.

1.3.12. Gram-Schmidt-ortogonalizálás.

```
> basis([z,y,x]);
```

$$[z, y, x] \quad (1.3.12.1)$$

```
> GramSchmidt([z,y,x]);
```

$$\left[[2, 3, 4], \left[\frac{31}{29}, -\frac{26}{29}, \frac{4}{29}\right], \left[\frac{4}{19}, \frac{4}{19}, -\frac{5}{19}\right]\right] \quad (1.3.12.2)$$

```
> L:=map(normalize,%);
```

$$L := \left[\left[\frac{2}{29}\sqrt{29} \quad \frac{3}{29}\sqrt{29} \quad \frac{4}{29}\sqrt{29}\right], \left[\frac{31}{1653}\sqrt{57}\sqrt{29} \quad -\frac{26}{1653}\sqrt{57}\sqrt{29} \quad \frac{4}{1653}\sqrt{57}\sqrt{29}\right], \left[\frac{4}{57}\sqrt{3}\sqrt{19} \quad \frac{4}{57}\sqrt{3}\sqrt{19} \quad -\frac{5}{57}\sqrt{3}\sqrt{19}\right]\right] \quad (1.3.12.3)$$

```
> map(norm,L,2);
```

$$[1, 1, 1] \quad (1.3.12.4)$$

▶ ***1.3.13. Következmény.***

▶ ***1.3.14. Következmény.***

▶ ***1.3.15. Tétel.***

▶ ***1.3.16. Az approximáció alapfeladata normált térben.***

▶ ***1.3.17. Legjobb lineáris approximáció belső szorzat térben.***

▶ ***1.3.18. Következmény: Bessel-egyenlőtlenség.***

▶ ***1.3.19. Ortogonális felbontási tétel.***

▶ ***1.3.20. Következmény.***

▶ ***1.3.21. Következmény.***

▶ ***1.3.22. Előállítási tétel.***

▶ ***1.3.23. Adjungált leképezés.***

▶ ***1.3.24. Tétel.***

▼ ***1.3.25. Adjungált operátor mátrixa.***

```
> a:=matrix([[1,2,I],[2,-I,3]]); htranspose(a);
```

$$a := \begin{bmatrix} 1 & 2 & I \\ 2 & -I & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & I \\ -I & 3 \end{bmatrix} \quad (1.3.25.1)$$

```
> d:=matrix(L); ds:=htranspose(d);
```

$$d := \begin{bmatrix} \frac{2}{29}\sqrt{29} & \frac{3}{29}\sqrt{29} & \frac{4}{29}\sqrt{29} \\ \frac{31}{1653}\sqrt{57}\sqrt{29} & -\frac{26}{1653}\sqrt{57}\sqrt{29} & \frac{4}{1653}\sqrt{57}\sqrt{29} \\ \frac{4}{57}\sqrt{3}\sqrt{19} & \frac{4}{57}\sqrt{3}\sqrt{19} & -\frac{5}{57}\sqrt{3}\sqrt{19} \end{bmatrix}$$

$$ds := \begin{bmatrix} \frac{2}{29}\sqrt{29} & \frac{31}{1653}\sqrt{57}\sqrt{29} & \frac{4}{57}\sqrt{3}\sqrt{19} \\ \frac{3}{29}\sqrt{29} & -\frac{26}{1653}\sqrt{57}\sqrt{29} & \frac{4}{57}\sqrt{3}\sqrt{19} \\ \frac{4}{29}\sqrt{29} & \frac{4}{1653}\sqrt{57}\sqrt{29} & -\frac{5}{57}\sqrt{3}\sqrt{19} \end{bmatrix} \quad (1.3.25.2)$$

```
> evalm(d&*ds); evalm(ds&*d);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.3.25.3)$$

1.3.26. Önadjungált és normális transzformációk.

```
> b:=matrix([[1,2,I],[2,3,-3*I],[-I,3*I,-1]]); bs:=htranspose
(b);
```

$$b := \begin{bmatrix} 1 & 2 & I \\ 2 & 3 & -3\,I \\ -I & 3\,I & -1 \end{bmatrix}$$

$$bs := \begin{bmatrix} 1 & 2 & I \\ 2 & 3 & -3\,I \\ -I & 3\,I & -1 \end{bmatrix} \quad (1.3.26.1)$$

```
> evalm(b&*bs); evalm(bs&*b);
```

$$\begin{bmatrix} 6 & 5 & -6\,I \\ 5 & 22 & -4\,I \\ 6\,I & 4\,I & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 & -6\,I \\ 5 & 22 & -4\,I \\ 6\,I & 4\,I & 11 \end{bmatrix} \quad (1.3.26.2)$$

```
> c:=matrix([[I,2,1],[-2,3*I,4],[-1,-4,0]]); cs:=htranspose
  (c);
```

$$c := \begin{bmatrix} I & 2 & 1 \\ -2 & 3\,I & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

$$cs := \begin{bmatrix} -I & -2 & -1 \\ 2 & -3\,I & -4 \\ 1 & 4 & 0 \end{bmatrix} \quad (1.3.26.3)$$

```
> evalm(c&*cs); evalm(cs&*c);
```

$$\begin{bmatrix} 6 & 4-8\,I & -8-I \\ 4+8\,I & 29 & 2-12\,I \\ -8+I & 2+12\,I & 17 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4-8\,I & -8-I \\ 4+8\,I & 29 & 2-12\,I \\ -8+I & 2+12\,I & 17 \end{bmatrix} \quad (1.3.26.4)$$

1.3.27. Példa.

```
> a:=matrix([[1,2,I],[2,-I,3],[-1,2*I,4]]); as:=htranspose(a)
  ;
```

$$a := \begin{bmatrix} 1 & 2 & I \\ 2 & -I & 3 \\ -1 & 2\,I & 4 \end{bmatrix}$$

$$as := \begin{bmatrix} 1 & 2 & -1 \\ 2 & I & -2\,I \\ -I & 3 & 4 \end{bmatrix} \quad (1.3.27.1)$$

```
> evalm(a&*as); evalm(as&*a);
```

$$\begin{bmatrix} 6 & 2+5\,I & -1 \\ 2-5\,I & 14 & 8 \\ -1 & 8 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2-4\,I & 2+I \\ 2+4\,I & 9 & -3\,I \\ 2-I & 3\,I & 26 \end{bmatrix} \qquad (1.3.27.2)$$

1.3.28.Polarizációs formula.

1.3.29. Következmény.

1.3.30. Következmény.

1.3.31. Tétel.

1.3.32. Következmény.

1.3.33. Megjegyzés.

```
> orthog(d); orthog(ds);
```

true

true (1.3.33.1)

1.3.34. Tétel.

1.3.35. Felső háromszög alak.

```
> a:=map(evalf,a); as:=map(evalf,as);
  eigenvects(as); e3:=normalize(op(1,%[1][3]));
```

$$a := \begin{bmatrix} 1. & 2. & 1.\,I \\ 2. & -1.\,I & 3. \\ -1. & 2.\,I & 4. \end{bmatrix}$$

$$as := \begin{bmatrix} 1. & 2. & -1. \\ 2. & 1.\,I & -2.\,I \\ -1.\,I & 3. & 4. \end{bmatrix}$$

$[-1.830172721 + 1.121842640\,I,\ 1,$
$\{[[-0.4108131960 + 0.4175601040\,I,$
$0.2289381566 - 0.6993571032\,I,$
$-0.2363592574 + 0.2439207700\,I]]\}],\ [4.395490368$
$-1.351578556\,I,\ 1,$

$\{[[0.2014302949 - 0.1949564944\,I,$

$0.4867552051 - 0.0799409069\,I,$

$0.5530548001 + 0.7743399524\,I]]\}], [2.434682347$

$+ 1.229735916\,I, 1,$

$\{[[-0.1293138255 - 0.8134486314\,I,$

$0.2451412887 - 0.4247678157\,I,$

$-0.3245201583 + 0.4765266198\,I]]\}]$

$e3 := [[-0.4108131960 + 0.4175601040\,I,$ (1.3.35.1)

$0.2289381566 - 0.6993571032\,I,$

$-0.2363592574 + 0.2439207700\,I]]$

```
> f11:=vector([1,0,0]); c1:=dotprod(f11,e3);
  g11:=normalize(evalm(f11-c1*e3));
```

$$f11 := [1 \quad 0 \quad 0]$$

$$c1 := -0.4108131960 - 0.4175601040\,I$$

$g11 := [[0.8104789186 + 0.\,I, 0.4763534642 - 0.2365387696\,I,$ (1.3.35.2)

$-0.2454734842 + 0.001865162632\,I]]$

```
> f12:=vector([0,1,0]); c1:=dotprod(f12,e3); c2:=dotprod(f12,
  g11);
  g12:=normalize(evalm(f12-c1*e3-c2*g11));
```

$$f12 := [0 \quad 1 \quad 0]$$

$$c1 := 0.2289381566 + 0.6993571032\,I$$

$$c2 := 0.4763534642 + 0.2365387696\,I$$

$g12 := [[0. + 0.\,I, 0.4190748738 + 0.\,I,$ (1.3.35.3)

$0.8162567697 + 0.3976193356\,I]]$

```
> a1:=matrix(2,2):a1[1,1]:=dotprod(multiply(a,g11),g11):
  a1[2,1]:=dotprod(multiply(a,g11),g12):
  a1[1,2]:=dotprod(multiply(a,g12),g11):
  a1[2,2]:=dotprod(multiply(a,g12),g12):
  print(a1);a1s:=htranspose(a1);
```

$$\begin{bmatrix} 2.173185067 - 0.8865758350\,I & 0.5435433324 + 1.007128936\,I \\ -0.4235426005 + 1.111037774\,I & 4.656987654 + 1.008418474\,I \end{bmatrix}$$

$a1s := \Big[[2.173185067 + 0.8865758350\,I,$ (1.3.35.4)

$-0.4235426005 - 1.111037774\,I], [$

0.5435433324 − 1.007128936 I, 4.656987654 − 1.008418474 I]

```
> eigenvects(a1s); f2:=op(1,%[1][3]);
  e2:=normalize(evalm(f2[1]*g11+f2[2]*g12));
```

[2.434682354 + 1.229735916 I, 1,

{[[−0.6623360963 + 0.6670597607 I,

0.1619977097 − 0.3001648099 I]]}], [4.395490367

− 1.351578555 I, 1,

{[[0.2724344358 − 0.2793541240 I,

0.9729764148 + 0.3530859180 I]]}]

f2 := [[−0.6623360963 + 0.6670597607 I,

0.1619977097 − 0.3001648099 I]]

e2 := [[−0.5368094431 + 0.5406378735 I, (1.3.35.5)

−0.08983142915 + 0.3486328633 I,

0.4129248338 − 0.3455789846 I]]

```
> f21:=vector([1,0,0]); c1:=dotprod(f21,e2); c2:=dotprod(f21,
  e3);
  e1:=normalize(evalm(f21-c1*e2-c2*e3));
```

f21 := [1 0 0]

c1 := −0.5368094431 − 0.5406378735 I

c2 := −0.4108131960 − 0.4175601040 I

e1 := [[0.2764459968 + 0. I, 0.5403151184 − 0.1921769406 I, (1.3.35.6)

0.7579923610 + 0.1419605178 I]]

```
> ta:=augment(e1,e2,e3); evalm(inverse(ta)&*a&*ta);
```

ta := [[0.2764459968 + 0. I, −0.5368094431 + 0.5406378735 I,

−0.4108131960 + 0.4175601040 I], [

0.5403151184 − 0.1921769406 I,

−0.08983142915 + 0.3486328633 I,

0.2289381566 − 0.6993571032 I], [

0.7579923610 + 0.1419605178 I,

0.4129248338 − 0.3455789846 I,

$-0.2363592574+0.2439207700\,I]$

$[4.395490369+1.351578555\,I,\ 0.981115142-1.177759536\,I,$ (1.3.35.7)

$-0.7172726961+0.6052215020\,I],\ [-2.\,10^{-9}-2.4\,10^{-9}\,I,$
$2.434682352-1.229735915\,I,\ 0.1199185482+2.135427372\,I],\ [$
$1.75\,10^{-9}+4.6\,10^{-9}\,I,\ 4.4\,10^{-9}+9.\,10^{-10}\,I,$

$-1.830172722-1.121842640\,I]$

1.3.36. *Segédtétel.*

1.3.37. *Normális transzformáció átlós alakja.*

```
> print(b); Lb:=evalf(eigenvects(b));
```

$$\begin{bmatrix} 1 & 2 & I \\ 2 & 3 & -3\,I \\ -I & 3\,I & -1 \end{bmatrix}$$

$Lb := [5.064969131-1.\,10^{-9}\,I,\ 1.,$ (1.3.37.1)
$\{[[1.,\ 2.590807476-7.958787212\,10^{-10}\,I,$
$8.976397949\,10^{-10}+1.116645820\,I]]\}],\ [-3.400399149$
$-5.660254040\,10^{-10}\,I,\ 1.,$
$\{[[1.,\ -1.225863560-7.92192835\,10^{-11}\,I,$
$-6.193515427\,10^{-10}+1.948672028\,I]]\}],\ [1.335430019$
$+1.166025404\,10^{-9}\,I,\ 1.,$
$\{[[1.,\ -0.1296497970+4.888495928\,10^{-10}\,I,$
$-2.34384875\,10^{-11}-0.5947296133\,I]]\}]$

```
> b1:=op(1,Lb[1][3]); b2:=op(1,Lb[2][3]); b3:=op(1,Lb[3][3]);
tb:=augment(normalize(b1),normalize(b2),normalize(b3));
```

```
evalm(inverse(tb)&*b&*tb);
```

$b1 := \left[[1., 2.590807476 - 7.958787212\ 10^{-10}\ I, 8.976397949\ 10^{-10} + 1.116645820\ I]\right]$

$b2 := \left[[1., -1.225863560 - 7.92192835\ 10^{-11}\ I, -6.193515427\ 10^{-10} + 1.948672028\ I]\right]$

$b3 := \left[[1., -0.1296497970 + 4.888495928\ 10^{-10}\ I, -2.34384875\ 10^{-11} - 0.5947296133\ I]\right]$

$tb := \left[[0.3340918169, 0.3984075083, 0.8541979365], [0.8655675769 - 2.658965680\ 10^{-10}\ I, -0.4883932465 - 3.156155735\ 10^{-11}\ I, -0.1107465891 + 4.175743134\ 10^{-10}\ I], [2.998941100\ 10^{-10} + 0.3730622308\ I, -2.467543049\ 10^{-10} + 0.7763655672\ I, -2.002110766\ 10^{-11} - 0.5080168085\ I]\right]$

$$\left[[5.064969132 + 0.\ I, -2.\ 10^{-9} + 5.480145087\ 10^{-10}\ I, -3.\ 10^{-10} + 1.375856464\ 10^{-9}\ I], [5.\ 10^{-10} + 8.716398007\ 10^{-10}\ I, -3.400399150 - 6.\ 10^{-19}\ I, 1.\ 10^{-9} + 8.922149470\ 10^{-10}\ I], [1.3\ 10^{-9} - 6.780241944\ 10^{-10}\ I, 1.1\ 10^{-9} - 5.771081405\ 10^{-10}\ I, 1.335430019 + 5.\ 10^{-19}\ I]\right] \qquad (1.3.37.2)$$

```
> print(c); evalf(eigenvects(c));
```

$$\begin{bmatrix} I & 2 & 1 \\ -2 & 3\,I & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

$$\left[-2.\,10^{-9} + 0.9126830441\,I,\ 1.,\ \{[[-3.704757828 + 1.690634827\,I,\ 1.,\ -1.852378915 + 0.3234881742\,I]]\}\right],\ \left[1.913397460\,10^{-9} + 6.360361017\,I,\ 1.,\ \{[[0.1208684493 - 0.384383483\,I,\ 1.,\ 0.06043422416 + 0.647898512\,I]]\}\right],\ \left[2.086602540\,10^{-9} - 3.273044063\,I,\ 1.,\ \{[[0.3080273189 + 0.5040934832\,I,\ 1.,\ 0.1540136594 - 1.316214274\,I]]\}\right] \quad (1.3.37.3)$$

```
> map(evalf,d); Ld:=eigenvects(%);
  abs(Ld[1][1]); abs(Ld[2][1]); abs(Ld[3][1]);
```

$$\begin{bmatrix} 0.3713906763 & 0.5570860147 & 0.7427813528 \\ 0.7624744001 & -.6394946585 & 0.09838379358 \\ 0.5298129430 & 0.5298129430 & -.6622661788 \end{bmatrix}$$

$$Ld := \left[-0.9651850825 - 0.2615678890\,I,\ 1,\ \{[[0.4651221388 + 0.4407628439\,I,\ -1.029175978 + 0.1082692855\,I,\ 0.09014493297 - 1.038110678\,I]]\}\right],\ \left[0.999999998,\ 1,\ \{[\,-.890383449\ \ -.4395241902\ \ -.4238805372\,]\}\right],\ \left[-0.9651850825 + 0.2615678890\,I,\ 1,\ \{[[0.4651221388 - 0.4407628439\,I,\ -1.029175978 - 0.1082692855\,I,\ 0.09014493297 + 1.038110678\,I]]\}\right]$$

$$1.000000002$$

$$0.999999998$$

$$1.000000002 \quad (1.3.37.4)$$

▶ ***1.3.38. Következmény.***

▶ ***1.3.39. Valós önadjungált transzformáció átlós alakja.***

1.3.40. Poláris felbontás.

*1.3.41. Szinguláris érték felbontás.

```
> with(LinearAlgebra);
```

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, (1.3.41.1)
BezoutMatrix, BidiagonalForm, BilinearForm,
CharacteristicMatrix, CharacteristicPolynomial, Column,
ColumnDimension, ColumnOperation, ColumnSpace,
CompanionMatrix, ConditionNumber, ConstantMatrix,
ConstantVector, Copy, CreatePermutation, CrossProduct,
DeleteColumn, DeleteRow, Determinant, Diagonal,
DiagonalMatrix, Dimension, Dimensions, DotProduct,
EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal,
ForwardSubstitute, FrobeniusForm, GaussianElimination,
GenerateEquations, GenerateMatrix, GetResultDataType,
GetResultShape, GivensRotationMatrix, GramSchmidt,
HankelMatrix, HermiteForm, HermitianTranspose,
HessenbergForm, HilbertMatrix, HouseholderMatrix,
IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal,
IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main,
LUDecomposition, LeastSquares, LinearSolve, Map, Map2,
MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse,
MatrixMatrixMultiply, MatrixNorm, MatrixPower,
MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial,
Minor, Modular, Multiply, NoUserValue, Norm, Normalize,
NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm,
QRDecomposition, RandomMatrix, RandomVector, Rank,
RationalCanonicalForm, ReducedRowEchelonForm, Row,
RowDimension, RowOperation, RowSpace, ScalarMatrix,
ScalarMultiply, ScalarVector, SchurForm, SingularValues,
SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix,
ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector,
VandermondeMatrix, VectorAdd, VectorAngle,

VectorMatrixMultiply, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

```
> a:=RandomMatrix(5,3); a:=evalf(a);
```

$$a := \begin{bmatrix} 81 & -29 & 20 \\ 89 & 9 & 39 \\ 92 & 81 & -35 \\ -2 & 35 & 26 \\ -46 & 80 & -74 \end{bmatrix}$$

$$a := \begin{bmatrix} 81. & -29. & 20. \\ 89. & 9. & 39. \\ 92. & 81. & -35. \\ -2. & 35. & 26. \\ -46. & 80. & -74. \end{bmatrix} \quad (1.3.41.2)$$

```
> u,s,v:=SingularValues(a,output=['U','S','Vt']);
```

u, *s*, *v* := [[−.515723273660289472, 0.132151019921274204, (1.3.41.3)

−.394622316425569053, 0.633450138422476439,
0.399474357688131332], [−.589733916134563164,
−0.0326996011735467204, 0.319526216340236868,
−.571601651692807411, 0.471507362717730971], [
−.455182309691849196, −.721659155238456695,
−.117883152916075678, 0.0264894891117678222,
−.507364758166423235], [−0.0292525854562412056,
−.108901523169650169, 0.847015017207565446,
0.519344796716199820, 0.0114580645892817488], [
0.422133585910225095, −.669936837268040542,
−.104140997372039301, 0.0398969047045862335,

0.600458790741895565]], [163.242561317803535, 143.317444973893259, 47.7909629591017620, 0., 0.],

[[−.952547636595486358, 0.0338480904927120698, −.302501746749255118], [−.192326737454134228, −.837215370932909875, 0.511938325127079263], [−.235930977356941879, 0.545824815708912170, 0.803997415717072283]]

```
> sm:=DiagonalMatrix(s[1..3],5,3); u.sm.v;
```

sm := [[163.242561317803535, 0, 0], [0, 143.317444973893259, 0], [0, 0, 47.7909629591017620], [0, 0, 0], [0, 0, 0]]

(1.3.41.4)

[81.0000000000000284, −29.0000000000000142, 20.], [89., 9., 39.0000000000000072], [92., 81.0000000000000426, −35.0000000000000072], [−2.00000000000000268, 35.0000000000000142, 26.0000000000000320], [−45.9999999999999858, 80.0000000000000568, −74.0000000000000142]

*1.3.42. Általánosított inverz.

*1.3.43. Feladat.

*1.3.44. Tétel.

```
> map(x->1/x,s[1..3]); sdagm:=DiagonalMatrix(%,3,5);
```

$$\begin{bmatrix} 0.006125853405 \\ 0.006977517636 \\ 0.02092445806 \end{bmatrix}$$

sdagm := [[0.006125853405, 0, 0, 0, 0], [0, 0.006977517636, 0, 0, 0], [0, 0, 0.02092445806, 0, 0]] (1.3.44.1)

```
> adag:=HermitianTranspose(v).sdagm.HermitianTranspose(u);
```

adag := [[0.00478013269218580551, 0.00190766368761816746, 0.00420646194983093167, −0.00386464705353093155, −0.00105007370269452399], [−0.00538593543529507398, 0.00371807754595812482, 0.00277497021375292973, 0.0103039363603375667, 0.00281168518073109845], [(1.3.44.2)

−0.00521108579869855419, 0.00635147673366757461,
−0.00371749206804140864, 0.0139147164699966654,
−0.00492728747005008547]]

```
> adag.a;
```

(1.3.44.3)

[[1.00000000008052292, $-3.76639108356363606\ 10^{-11}$, $2.43325873772876067\ 10^{-11}$], [$-3.76640112327575327\ 10^{-11}$, 0.99999999992147028, $9.97882974168584624\ 10^{-11}$], [$2.43322566956249986\ 10^{-11}$, $9.97885957892963304\ 10^{-11}$, 1.00000000002413336]]

1.3.31. Tétel: QR-felbontás.

```
> q,r:=QRDecomposition(a);
```

$q, r :=$ [[−.511614095472075370, 0.294374456096020809, (1.3.45.1)
−.301257127043764339], [−.562143882679193907,
−0.0110671115747957953, 0.367184058603824304], [
−.581092552881863234, −.598350990052114229,
−.214911253372684186], [0.0126324468017796377,
−.287917987852580381, 0.804421126300396527], [
0.290546276440931617, −.687241241205836739,

$-.284850513862875687]$, $[[-158.322455766704167,$

$-13.6051452055166653, -32.9896348228475204], [0.,$
$-122.159322296486266, 69.7681405873921960], [0., 0.,$

$57.8108169223809085]]$

```
> q.r;
```

$$[[81.0000000000000142, -29., 20.], [89.0000000000000142, 9.00000000000001066, 39.0000000000000072], [92., 80.9999999999999858, -35.], [-2., 35.0000000000000072, 26.0000000000000072], [-46., 80.0000000000000142, -74.]] \quad (1.3.45.2)$$

```
> q,r:=QRDecomposition(a,fullspan);
```

$$q, r := [[-.511614095472075370, 0.294374456096020809, \quad (1.3.45.3)$$

−.301257127043764339, 0.358731372050832941,
0.657381657048876300], [−.562143882679193907,
−0.0110671115747957953, 0.367184058603824304,
−.728858280178455443, 0.133466283121541435], [
−.581092552881863234, −.598350990052114229,
−.214911253372684186, 0.270210314472837276,
−.430240719615387068], [0.0126324468017796377,
−.287917987852580381, 0.804421126300396527,
0.448024204168981865, 0.262915609272393269], [
0.290546276440931617, −.687241241205836739,
−.284850513862875687, −.257561853839173960,
0.543879043382951455]], [[−158.322455766704167,
−13.6051452055166653, −32.9896348228475204], [0.,
−122.159322296486266, 69.7681405873921960], [0., 0.,
57.8108169223809085], [0., 0., 0.], [0., 0., 0.]]

```
> q.r;
```

(1.3.45.4)

[[81.0000000000000142, −29., 20.], [89.0000000000000142,
9.00000000000001066, 39.0000000000000072], [92.,

$80.9999999999999858, -35.], [-2., 35.0000000000000072,$

$26.0000000000000072], [-46., 80.0000000000000142, -74.]]$

**1.3.46. Megjegyzés.*

**1.3.47. Feladat.*

```
> h:=HilbertMatrix(10); MatrixInverse(h);
```

$h := [[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}], [\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9},$

$\frac{1}{10}, \frac{1}{11}], [\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}], [\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8},$

$\frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}], [\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}], [\frac{1}{6},$

$\frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}], [\frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13},$

$\frac{1}{14}, \frac{1}{15}, \frac{1}{16}], [\frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}], [\frac{1}{9}, \frac{1}{10},$

$\frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}], [\frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15},$

$\frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}]\Bigg]$

(1.3.47.1)

(1.3.47.1)

[100, −4950, 79200, −600600, 2522520, −6306300, 9609600, −8751600, 4375800, −923780], [−4950, 326700, −5880600, 47567520, −208107900, 535134600, −832431600, 770140800, −389883780, 83140200], [79200, −5880600, 112907520, −951350400, 4281076800, −11237826600, 17758540800, −16635041280, 8506555200, −1829084400], [−600600, 47567520, −951350400, 8245036800, −37875637800, 101001700800, −161602721280, 152907955200, −78843164400, 17071454400], [2522520, −208107900, 4281076800, −37875637800, 176752976400, −477233036280, 771285715200, −735869534400, 382086104400, −83223340200], [−6306300, 535134600, −11237826600, 101001700800, −477233036280, 1301544644400, −2121035716800, 2037792556800, −1064382719400, 233025352560], [9609600, −832431600, 17758540800, −161602721280, 771285715200, −2121035716800, 3480673996800, −3363975014400, 1766086882560, −388375587600], [−8751600, 770140800, −16635041280, 152907955200, −735869534400, 2037792556800, −3363975014400, 3267861442560, −1723286307600, 380449555200], [4375800, −389883780,

8506555200, −78843164400, 382086104400, −1064382719400,
1766086882560, −1723286307600, 912328045200,
−202113826200], [−923780, 83140200, −1829084400,
17071454400, −83223340200, 233025352560, −388375587600,

380449555200, −202113826200, 44914183600]

```
> h:=HilbertMatrix(15);
```

$$h := \begin{bmatrix} \text{15 x 15 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \quad (1.3.47.2)$$

```
> x0:=Vector([1$i=1..15]);
```

$$x0 := \begin{bmatrix} \text{1 .. 15 } \mathit{Vector}_{column} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \quad (1.3.47.3)$$

```
> b:=h.x0; b[1..10]; b[11..15];
```

$$b := \begin{bmatrix} \text{1 .. 15 } \mathit{Vector}_{column} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1195757}{360360} \\ \frac{1715839}{720720} \\ \frac{23763863}{12252240} \\ \frac{6786821}{4084080} \\ \frac{113634299}{77597520} \\ \frac{101994671}{77597520} \\ \frac{30918957}{25865840} \\ \frac{28399557}{25865840} \\ \frac{604691361}{594914320} \\ \frac{5070400799}{5354228880} \end{bmatrix}$$

$$\begin{bmatrix} \frac{23745735331}{26771144400} \\ \frac{22341654331}{26771144400} \\ \frac{63306748493}{80313433200} \\ \frac{59997128993}{80313433200} \\ \frac{236266661971}{332727080400} \end{bmatrix} \tag{1.3.47.4}$$

```
> b:=evalf(b); h:=evalf(h);
```

$$b := \begin{bmatrix} \textit{1 .. 15 Vector}_{column} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$h := \begin{bmatrix} \textit{15 x 15 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \quad (1.3.47.5)$$

```
> x:=LinearSolve(h,b); x[1..10]; x[11..15];
```

$$x := \begin{bmatrix} \textit{1 .. 15 Vector}_{column} \\ \textit{Data Type: float}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$\begin{bmatrix} 0.999788450825984220 \\ 1.01724274308992757 \\ 0.648711803886287908 \\ 4.10252173386348762 \\ -13.5889316329352994 \\ 40.8826883097272074 \\ -63.4443468180658812 \\ 58.9328730427461024 \\ -24.6681889965934609 \\ 19.3985665680719883 \end{bmatrix}$$

$$\begin{bmatrix} -42.3403573630075627 \\ 44.1956655193574477 \\ -7.41206437348795610 \\ -9.56405531676626453 \\ 5.84003259969147148 \end{bmatrix} \quad (1.3.47.6)$$

```
> u,s,v:=SingularValues(h,output=['U','S','Vt']); s[1..10]; s
```

```
[11..15];
```

$u, s, v := \begin{bmatrix} \textit{15 x 15 Matrix} \\ \textit{Data Type: } float_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}, \begin{bmatrix} \textit{1 .. 15 } Vector_{column} \\ \textit{Data Type: } float_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$, [[*15 x 15 Matrix*], [*Data Type:* $float_8$], [*Storage: rectangular*], [*Order: Fortran_order*]]

$$\begin{bmatrix} 1.84592774617273059 \\ 0.426627957000250135 \\ 0.0572120925540026415 \\ 0.00563983471864011685 \\ 0.000436476611408183454 \\ 0.0000271085275106774550 \\ 0.00000136156531902204499 \\ 5.53016609757566885\ 10^{-8} \\ 1.81143420152069102\ 10^{-9} \\ 4.12472018236365762\ 10^{-11} \end{bmatrix} \begin{bmatrix} 8.74760906275593120\ 10^{-12} \\ 8.51326057707484599\ 10^{-12} \\ 4.89367543903241146\ 10^{-12} \\ 2.81276023743357307\ 10^{-12} \\ 2.14418317893619637\ 10^{-12} \end{bmatrix} \quad (1.3.47.7)$$

```
> map(x->1/x,s[1..9]); sdagm:=DiagonalMatrix(%,15,15);
```

$$\begin{bmatrix} 0.5417330132 \\ 2.343962658 \\ 17.47882232 \\ 177.3101606 \\ 2291.073505 \\ 36888.76128 \\ 7.344487892\ 10^{5} \\ 1.808263951\ 10^{7} \\ 5.520487572\ 10^{8} \end{bmatrix}$$

$$sdagm := \begin{bmatrix} \textit{15 x 15 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: diagonal} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{1.3.47.8}$$

```
> hdag:=HermitianTranspose(v).sdagm.HermitianTranspose(u);
```

$$hdag := \begin{bmatrix} \textit{15 x 15 Matrix} \\ \textit{Data Type: } float_{8} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{1.3.47.9}$$

```
> x:=hdag.b; x[1..10]; x[11..15];
```

$$x := \begin{bmatrix} \textit{1 .. 15 } Vector_{column} \\ \textit{Data Type: } float_{8} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix}$$

$$\begin{bmatrix} 1.00000868207093419 \\ 0.99965041420318812 0 \\ 1.00334094631489279 \\ 0.988106061071448494 \\ 1.01525288877746789 \\ 1.00141393598323702 \\ 0.989774210116593168 \\ 0.993305517913540825 \\ 1.00289546596650325 \\ 1.00835500565153779 \end{bmatrix}$$

$$\begin{bmatrix} 1.00612688068395073 \\ 0.998863099594927917 \\ 0.992460173183644656 \\ 0.993289312227716437 \\ 1.00715814682553173 \end{bmatrix} \quad (1.3.47.10)$$

*1.3.48. Feladat.

```
> a:=Matrix([[1.,0.],[0.,0.],[0.,0.]]); b:=Vector([1.,2.,3.])
  ;
```

$$a := \begin{bmatrix} 1. & 0. \\ 0. & 0. \\ 0. & 0. \end{bmatrix}$$

$$b := \begin{bmatrix} 1. \\ 2. \\ 3. \end{bmatrix} \quad (1.3.48.1)$$

```
> u,s,v:=SingularValues(a,output=['U','S','Vt']);
```

$$u, s, v := \begin{bmatrix} 1. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 1. \end{bmatrix}, \begin{bmatrix} 1. \\ 0. \\ 0. \end{bmatrix}, \begin{bmatrix} 1. & 0. \\ 0. & 1. \end{bmatrix} \quad (1.3.48.2)$$

```
> sdag:=map(x->1/x,s[1..1]); sdagm:=DiagonalMatrix(sdag[1..1]
  ,2,3);
```

$$sdag := [1.]$$

$$sdagm := \begin{bmatrix} 1. & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.3.48.3)$$

```
> adag:=HermitianTranspose(v).sdagm.HermitianTranspose(u);
```

$$adag := \begin{bmatrix} 1. & 0. & 0. \\ 0. & 0. & 0. \end{bmatrix} \quad (1.3.48.4)$$

```
> x:=adag.b;
```

$$x := \begin{bmatrix} 1. \\ 0. \end{bmatrix} \quad (1.3.48.5)$$

```
> a[2,2]:=1.*10^(-12); a;
```

$$a_{2,\,2} := 1.000000000\ 10^{-12}$$

$$\begin{bmatrix} 1. & 0. \\ 0. & 1.000000000\ 10^{-12} \\ 0. & 0. \end{bmatrix} \quad (1.3.48.6)$$

```
> u,s,v:=SingularValues(a,output=['U','S','Vt']);
```

$$u,\ s,\ v := \begin{bmatrix} 1. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 1. \end{bmatrix}, \begin{bmatrix} 1. \\ 9.9999999999999998\ 10^{-13} \\ 0. \end{bmatrix}, \begin{bmatrix} 1. & 0. \\ 0. & 1. \end{bmatrix} \quad (1.3.48.7)$$

```
> sdag:=map(x->1/x,s[1..2]); sdagm:=DiagonalMatrix(sdag[1..2]
,2,3);
```

$$sdag := \begin{bmatrix} 1. \\ 1.000000000\ 10^{12} \end{bmatrix}$$

$$sdagm := \begin{bmatrix} 1. & 0 & 0 \\ 0 & 1.000000000\ 10^{12} & 0 \end{bmatrix} \quad (1.3.48.8)$$

```
> adag:=HermitianTranspose(v).sdagm.HermitianTranspose(u);
```

$$adag := \begin{bmatrix} 1. & 0. & 0. \\ 0. & 1.0000000000000000\ 10^{12} & 0. \end{bmatrix} \quad (1.3.48.9)$$

```
> x:=adag.b;
```

$$x := \begin{bmatrix} 1. \\ 2.0000000000000000\ 10^{12} \end{bmatrix} \quad (1.3.48.10)$$

```
>
```

▶ **1.3.49. Tyihonov-regularizáció.*

► **1.3.50. Feladat.*

► 2. Többváltozós függvények

► 3. Függvénysorozatok és függvénysorok